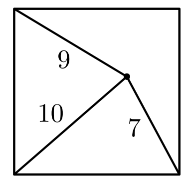
Solutions to February 2011 Problems

Problem 1. The distances from a point inside a square to 3 consecutive vertices of the square are 9, 10, and 7 as shown. Find (exactly) the area of the square.



Solution. Let x be the side of the square. Look first at the triangle with sides 9, x, and 10. If θ is the angle opposite the side of length 9, we have, by the Cosine Law,

$$81 = x^2 + 100 - 20x\cos\theta.$$

Now look at the triangle with sides 7, 10, and x. The cosine of the angle opposite the side of length 7 is $\sin\theta$, so again by the Cosine Law,

$$49 = x^2 + 100 - 20x\sin\theta.$$

Our two equations can be rewritten as

$$20x\cos\theta = x^2 + 19 \quad \text{and} \quad 20x\sin\theta = x^2 + 51.$$

Eliminate θ by squaring both sides of the above equations and adding. We obtain

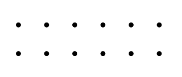
$$400x^2 = (x^2 + 19)^2 + (x^2 + 51)^2.$$

After some mild simplification we arrive at $x^4 - 130x^2 + 1481 = 0$. Then

$$x^2 = 65 \pm 14\sqrt{14}.$$

It is clear that $x^2 = 65 - 14\sqrt{14}$ is geometrically impossible. So we conclude that the area of our square is $65 + 14\sqrt{14}$.

Problem 2. Some of the 21 dots are coloured blue, and the rest red. Show that there are 4 dots, all of the same colour, which are the vertices of a rectangle with horizontal and vertical edges.



Solution. Since there are 21 dots, at least 11 are red, or at least 11 are blue. Without loss of generality we may assume that at least 11 are blue. We will prove a somewhat stronger result. Suppose that exactly 11 are blue, and the rest uncoloured. Then there are 4 blue dots which are the vertices of a rectangle with horizontal and vertical edges.

One row has at least as many blue dots as any of the other rows. Without loss of generality we may assume that this is the top row. Then the top row has (a) 7 blue dots or (b) 6 blue dots or (c) 5 blue dots or (d) 4 blue dots.

Case (a): There there are 4 other blue dots. So one of the remaining rows, without loss of generality the second row, has at least 2 blue dots. Take 2 of these blue dots, and the 2 blue dots immediately above them. These 4 blue dots are the vertices of a rectangle.

Case (b): To help in visualizing things, we can assume that the first 6 dots in the top row are blue, and the last is uncoloured. There is a total of 5 blue dots in the next two rows.. So one of the remaining rows, without loss of generality the second row, has at least 3 blue dots. At most 1 of these blue dots is immediately below the uncoloured dot in the top row, so there are at least 2 blue dots in the second row that are immediately below blue dots in the top row. Thus again we have found 4 blue dots that are the vertices of a rectangle.

Case (c): We can assume that the first 5 top row dots are blue. There is a total of 6 blue dots in the next two rows. If some row other than the top row (say the second row) has 4 or more blue dots, then essentially the same argument as in the first two cases settles thins. Now suppose that the second and third row each have 3 blue dots. If in any one of these two rows there are 2 blue dots that are below top row blue dots, again we are finished. So the only situation that could cause a problem occurs if in both the second and third row, 2 of the blue dots are under uncoloured top row dots. But then these 4 second and third row blue dots are the vertices of a rectangle.

Case (d): We can assume that the first 4 dots of the top row are blue, and that of the remaining 7 blue dots, 4 are in the second row and 3 in the third. If 2 or more second row blue dots are below top row blue dots, we are finished. So we can assume that beneath each uncoloured top row dot, there is a second row blue dot. We are also finished unless at least 2 of the third row blue dots are under uncoloured top row dots. But then immediately above these third row blue dots there are second row blue dots, so again we get a rectangle with blue vertices.

Comment. It is not hard to find a two-colouring of the 18 dots of a 3×6 array (9 will be red and 9 blue) for which there is no rectangle all of whose vertices are the same colour. So the smallest n for which a two-colouring of the $3 \times n$ array forces a monochromatic rectangle is given by $n = 7$.

Problem 3. Find all integers n such that $2^n - 15$ is a perfect square. Of course, proof is needed that the list is complete.

Solution. It is easy to see that $n = 4$ and $n = 6$ work. We will show that nothing else does.

Look first at even values of n. Let $n = 2m$, where m is an integer, and let $x = 2^m$. We are looking for solutions of $x^2 - 15 = y^2$, with y an integer and x a power of 2. Equivalently, we want to solve the equation

$$(x + y)(x - y) = 15,$$

with x a power of 2, and therefore positive. Without loss of generality $y \geq 0$. Thus $x + y = 15$, $x - y = 1$, or $x + y = 5$, $x - y = 3$. The first possibility yields $x = 8$, and the second yields $x = 4$. Each happens to be a power of 2. If $x = 8$, we have $m = 3$, and hence $n = 6$. If $x = 4$ then $n = 4$. Thus the only solutions with n even are $n = 6$ and $n = 4$.

Finally, we show that there are no solutions with n odd. If n is odd, then the remainder when 2^n is divided by 3 is equal to 2, so the remainder when $2^n - 15$ is divided by 3 is also equal to 2. But any perfect square leaves a remainder of 0 or 1 on division by 3. So if n is odd then $2^n - 15$ cannot be a perfect square.

There are other possible congruential arguments, using for example the remainder on division by 5, or more familiarly by looking at units digits (remainder on division by 10). Note that if n is odd then the units digit of 2^n is 2 or 8, so the units digit of $2^n - 15$ is 7 or 3. But the units digit of a perfect square must be one of 0, 1, 4, 5, 6, or 9. Hence if n is odd then $2^n - 15$ cannot be a perfect square.

Problem 4. How many ordered triples (x, y, n) are there such that x and y are positive integers, n is an integer greater than 1, and $x^n - y^n = 2^{144}$?

Solution. We first examine the possibility $n = 2$. So we want to count the positive solutions of

$$(x + y)(x - y) = 2^{144}.$$

If (x, y) is any such solution, let $x + y = a$ and $x - y = b$. Then $ab = 2^{144}$ and $a > b$. Moreover, a and b must be both even.Their product is even, so at least one is even. But it is easy to see that ofr any integers x and y, the numbers $x + y$ and $x - y$ have the same parity (are both odd or both even).

Conversely, let a and b be positive integers, both even, such that $ab = 2^{144}$ and $a > b$. Set $x + y = a$ and $x - y = b$. Then $x = (a + b)/2$ and $y = (a - b)/2$. Note that since a and b are both even, then x and y are integers. It is clear that $x^2 - y^2 = 2^{144}$.

It follows that the number of positive integer solutions of $x^2 - y^2 = 2^{144}$ is precisely the same as the number of representations of 2^{144} as ab, where $a > b$ and $b \neq 1$. The possibilities for b are 2^1, 2^2, 2^3, and so on up to 2^{71}. We conclude that there are 71 ordered triples $(x, y, 2)$ such that x and y are positive integers and $x^2 - y^2 = 2^{144}$.

Next we show that there are no solutions with n odd (and greater than 1). For suppose to the contrary that $x^n - y^n = 2^{144}$, and that n is odd and greater than 1. Let 2^e be the greatest power of 2 that divides both x and y, and let $x = 2^e u$ and $y = 2^e v$. Then $u^n - v^n = 2^{144-ne}$, a power of 2, and u and v are positive, with at least one of u and v odd. Now use the factorization

$$u^n - v^n = (u - v)(u^{n-1} + u^{n-2}v + \cdots + v^{n-1}).$$

Note that the term $Q(u, v) = u^{n-1} + u^{n-2}v + \cdots + v^{n-1}$ is greater than 1. Also, if one of u or v is even, the other is odd, so $Q(u, v)$ is odd. And if u and v are both odd, then $Q(u, v)$ is a sum of an odd number of odd terms, so it is odd. We conclude that $u^n - v^n$ has an odd factor greater than 1, so cannot be a power of 2. It follows that $x^n - y^n$ cannot be a power of 2.

Next we show that n cannot be twice an odd number (and greater than 2). For let $n = 2m$, where m is an odd number greater than 1. Suppose that $x^n - y^n = 2^{144}$, where x and y are positive. Let $s = x^2$, $t = y^2$. Then $s^m - t^m = 2^{144}$. But by the previous argument, that is impossible.

Finally, we show that n cannot be a multiple of 4. For suppose to the contrary that $n = 4m$, and $x^{4m} - y^{4m} = 2^{144}$. Let $s = x^m$ and $t = y^m$. Then $s^4 - t^4 = 2^{144}$, and therefore

$$(s - t)(s + t)(s^2 + t^2) = 2^{144}.$$

It follows that $s - t$, $s + t$, and $s^2 + t^2$ are all powers of 2. This is impossible. For note that

$$(s - t)^2 + (s + t)^2 = 2(s^2 + t^2).$$

So if $s - t$, $s + t$, and $s^2 + t^2$ are all powers of 2, then $(s - t)^2 + (s + t)^2$ is a power of 2. But it is easy to show that the sum $a + b$ of two powers of 2 is a power of 2 if and only if $a = b$. However, if $s - t = s + t$, then $s = t$, and our product would be 0, which is not a power of 2.

© 2011 by Andrew Adler
http://www.pims.math.ca/education/math_problems/
http://www.math.ubc.ca/~adler/problems/