Solutions to February 2012 Problems

Problem 1. A certain curve consist of all points (u, v) such that there are two tangent lines to the curve $y = x^2$ that pass through (u, v), and these tangent lines are perpendicular to each other. Find an equation for the curve.

Solution. The 'generic" line with slope m has equation of the shape $y = mx + b$. This is tangent to the curve $y = x^2$ if it meets the curve at two coinciding points.

The line meets the curve at the points whose x-coordinates satisfy the equation

$$x^2 = mx + b,$$

or equivalently $x^2 - mx - b = 0$. This has two equal roots precisely if $m^2 + 4b = 0$. So the equation of the line has shape

$$y = mx - \frac{m^2}{4}.$$

If a line is perpendicular to one with slope m, it has equation of the shape $y = -\frac{1}{m}x + c$. We have tangency precisely if the equation is

$$y = -\frac{1}{m}x - \frac{1}{4m^2}.$$

The point (u, v) lies on both lines precisely if it satisfies both equations, that is, if

$$v = mu - \frac{m^2}{4} \qquad \text{and} \qquad v = -\frac{1}{m}u - \frac{1}{4m^2}.$$

Now what? It might be nice to use one equation to express m in terms of u and v, and then substitute in the other equation. This is a little messy. It is easier to eliminate u, or v. It turns out that eliminating u gets us where we want a little faster. So multiply both sides of the equation $v = (-1/m)u - 1/(4m^2)$ by m^2. We get

$$m^2 v = -mu - \frac{1}{4}.$$

From $v = mu = \frac{m^2}{4}$, we obtain, by adding, that

$$(1 + m^2)v = -\frac{1}{4}(1 + m^2)$$

Which simplifies spectacularly to $v = \dfrac{-1}{4}$. So the "curve" is a horizontal line. (We could after some experimentation have decided that this was likely to be the answer. To *verify* that it is correct would involve quite a bit less work.)

Problem 2. How many different 12-letter "words" can be made using 3 A's, 4 B's, and 5 C's which have no consecutive occurrences of the letter B?

Solution. First we count the number of 8-letter words made up solely of $A'sand$C'x. Such a word is completely determined once we know the location of the A's, so there are $\binom{8}{3}$ of them. Take *any* such word w. The B's can be placed in any of the 9 "gaps" determined by w, where the space immediately to the left of the first letter of w, and the space immediately to the right of the last letter of w, count as a gap. We must *choose* 5 of these gaps. That can be done in $\binom{9}{5}$ ways. Thus the total number of words with the desired property is

$$\binom{8}{3}\binom{9}{5}.$$

Problem 3. Find (with proof) all quadruples of consecutive odd integers whose product is a perfect square.

Solution. It was not specified that the integers are positive. So an easy example is the quadruple $(-3, -1, 1, 3)$. Are there any others? Can't think of any, but that is not a proof that there aren't any.

In problem solving, symmetry can be very helpful. So let the "average" of our four numbers be n. Then our four numbers are $n-1$, $n+1$, $n-3$, and $n+3$. Their product is $(n^2-1)(n^2-9)$, which is $n^4 - 10n^2 + 9$. By completing the square, we can see that our product is equal to $(n^2-5)^2 - 16$. Can this be a perfect square?

Suppose that it is the perfect square x^2. Then $(n^2-5)^2 = x^2 + 16$. For what integer values of x is $x^2 + 16$ a perfect square? There are the obvious solutions $x^2 = 0$ and $x^2 = 9$. These give respectively $n^2 - 5 = \pm 4$ and $n^2 - 5 = \pm 5$. The cases $n^2 - 5 = \pm 4$ give an odd n, which makes our four numbers even, not odd. The case $n^2 - 5 = 5$ gives a non-integer n, while the case $n^2 - 5 = -5$ gives $n = 0$, which gives us the already found quadruple $(-3, -1, 1, 3)$.

The solution is not yet complete, for we must *prove* that the only (non-negative) x such that $x^2 + 16$ is a perfect square are $x = 0$ and $x = 3$. This is not hard, for the consecutive perfect squares are 0, 1, 4, 9, 16, 25, 36, 49, and so on. So quickly the gap between consecutive perfect squares is greater than 16, and therefore squares beyond 25 cannot be expressed as $x^2 + 16$.

Problem 4. Suppose there is a number M such that for all n,

$$|f(x_1 + x_2 + \cdots + x_n) - f(x_1) - f(x_2) - \cdots - f(x_n)| < M.$$

Show that $f(x+y) = f(x) + f(y)$ for all real numbers x and y.

Solution. We gradually get some information about f. Let $f(0) = a$. Put all the x_i equal to 0. We get that

$$|a - na| < M$$

for all n. This says in particular that for every positive integer n, we have $(n-1)|a| < M$. It follows that $a = 0$.

Now consider even n, say $n = 2m$. Let $x_i = x$ when i is odd, and let $x_i = -x$ when i is even. Se have $f(x + (-x) + x + (-x) + \cdots + x + (-x)) = f(0) = 0$. It follows that

$$m|(f(x) + f(-x)| < M$$

for all m. That forces $f(x) + f(-x) = 0$, and therefore $f(-x) = -f(x)$.

Finally, let n be a multiple of 3, say $n = 3m$. Let $x_1 = x + y$, $x_2 = -x$, $x_3 = -y$, and continue that way, alternating $x + y$, $-x$, and $-y$. Then

$$f(x_1 + x_2 + x_3 + \cdots + x_{3m-1} + x_{3m}) = f(0) = 0.$$

But $f(x_1) + f(x_2) + f(x_3) = f(x + y) + f(-x) + f(-y) = f(x + y) - f(x) - f(y)$, and therefore

$$f(x_1) + f(x_2) + f(x_3) + \cdots + f(x_{3m-2}) + f(x_{3m-1}) + f(x_{3m}) = m(f(x + y) - f(x) - f(y)).$$

It follows that

$$m|f(x + y) - f(x) - f(y)| < M$$

for all m, which implies that $f(x+y) - f(x) - f(y) = 0$, meaning that $(x+y) = f(x)+f(y)$.

Comment. The functional equation $f(x + y) = f(x) + f(y)$ is a famous one, usually called the Cauchy functional equation. It shows up naturally in a number of areas of mathematics.

It is not very hard to show that any *continuous* function f that satisfies the Cauchy functional equation has the shape $f(x) = kx$, where k is a constant. And it is in fact not possible to exhibit explicitly a solution that is not of this shape. However, using the *Axiom of Choice,* one can prove that there *are* solutions not of the shape kx. They are all very weird. (That's not quite a technical term.)

http://www.pims.math.ca/education/math_problems/
http://www.math.ubc.ca/~adler/problems/