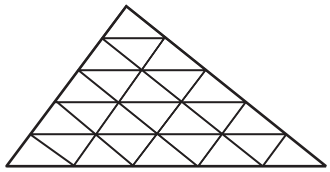
Solutions to March 2007 Problems

Problem 1. Show how to divide any triangle $\mathcal{T}$ into 2007 triangles similar to $\mathcal{T}$.

Solution. If we divide each side of a triangle into k equal parts, and through the division points draw lines parallel to the sides, the lines divide the triangle into k^2 small triangles each similar to the original. The illustration below has $k = 5$, but the procedure is perfectly general.



Take now $k = 44$. So we have divided $\mathcal{T}$ into 1936 congruent triangles each similar to $\mathcal{T}$. We need to make 71 more triangles. Take one of our daughter triangles, and divide it into 64 congruent triangles each similar to it, and therefore to $\mathcal{T}$ ($k = 8$). We have therefore added 63 triangles, and need 8 more. That can be done by taking one of our triangles and dividing it into 9 triangles.

Or else divide $\mathcal{T}$ into 43^2 triangles. We need 158 more. Divide one of the triangles into 12^2 triangles. We need 15 more. Divide any of the triangles into 4^2 triangles.

Or else divide $\mathcal{T}$ into 42^2 triangles. We need 243 more. Note that $243 = (3)(81)$. Take 81 of the daughter triangles, divide each of them into 4 triangles, and we are finished.

There many other ways to do the job. It should be clear that except for a few "small" values of n, we can divide any triangle $\mathcal{T}$ into n parts each similar to $\mathcal{T}$. The details might be interesting to work out—or not.

Problem 2. Find all x that satisfy the inequality

$$\left(\frac{\sqrt{x+1}-1}{x}\right)^2 > x+1.$$

Solution. It is worthwhile to look for shortcuts before starting to calculate, for tricks can be useful, specially in problems manufactured by tricky people. In particular, when we see something like $\sqrt{x+1}-1$, we should wonder whether its companion $\sqrt{x+1}+1$ might be helpful—and here it is.

Multiply "top" and "bottom" of $(\sqrt{x+1}-1)/x$ by $\sqrt{x+1}+1$ and simplify. We get $1/(\sqrt{x+1}+1)$. Write u for $\sqrt{x+1}$.

Our inequality becomes $1/(u+1)^2 > u^2$. Because u is non-negative, we can take square roots to obtain the equivalent inequality $1/(u+1) > u$, which in turn is equivalent to $u^2 + u - 1 < 0$.

The equation $u^2 + u - 1 = 0$ has roots $u = (-1 \pm \sqrt{5})/2$, and $u^2 + u - 1$ is negative only between these roots. But $u \geq 0$, so $0 \leq \sqrt{x+1} < (-1+\sqrt{5})/2$, or equivalently $-1 \leq x < (1-\sqrt{5})/2$.

Comment. In school work, sometimes an answer that looks like, say, $1/(\sqrt{7}-2)$ is considered unacceptable, since it has a square root in the denominator. One is expected to multiply "top" and "bottom" by $\sqrt{7}+2$, in order to "rationalize" the denominator, to transform the answer to $(\sqrt{7}+2)/3$.

Once upon a time, this made a certain amount of sense. In the old days (BC, before calculators), approximate numerical evaluation of $(\sqrt{7}-2)/3$ was far easier than approximate numerical evaluation of $1/(\sqrt{7}-2)$, so the first expression was considered to be simpler than the second, and one was expected to rationalize the denominator. But with a calculator, approximate evaluation of $1/(\sqrt{7}-2)$ is slightly simpler, in the sense that it involves fewer key strokes. Note that in our problem, rationalizing the *numerator* is the right strategy.

Problem 3. Download speed varies. Your browser posts a running estimate of the time needed to finish the download job. It does so by assuming that the *average* download speed so far will continue. The browser was downloading a large file. After 2 minutes, it estimated that there were 30 seconds left. For the next 5 minutes, the estimated remaining time to finish the job stayed at 30 seconds. What fraction of the file had been downloaded after these 5 minutes? (Assume that the program continued to function normally.)

Solution. Let the file be of size 1, and let $f(t)$ be the fraction of the file that had been downloaded by time t, where t is the time, in minutes, after the beginning of the download.

Since the file has size 1, the part that remains to be downloaded at time t is $1-f(t)$, and the average download speed up to time t is $f(t)/t$.

For $2 \le t \le 5$, the estimated remaining time is 0.5 minutes. It follows that

$$\frac{1-f(t)}{f(t)/t} = 0.5$$

Now solve for $f(t)$. A little manipulation shows that if $2 \le t \le 5$, then

$$f(t) = \frac{t}{t+0.5}.$$

Thus $f(5) = 5/(5+0.5) = 10/11$, and the fraction of the file that remains to be downloaded at time 5 is $1-10/11$, that is, $1/11$.

Problem 4. Find the maximum value taken on by $3x+4y$ as (x,y) ranges over all ordered pairs such that $x^2+y^2 \le 1$.

Solution. We will be close to the answer as soon as we have a clear grasp of the geometry. The points (x,y) with $x^2+y^2 \le 1$ fill up the disk with center the origin O and radius 1.

For any fixed k, $3x+4y=k$ is the equation of a line with slope $-3/4$. As k increases, the line moves upwards and to the right (see Figure 1). Thus the largest k such that the line $3x+4y=k$ meets the disk is the k for which the line is tangent to $x^2+y^2=1$ (and "above" it).

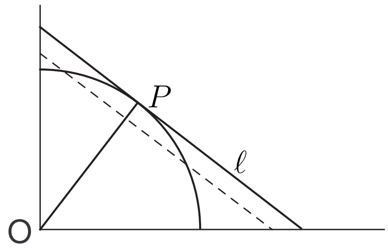


Figure 1: Maximizing $3x + 4y$ on the Unit Disk

Let ℓ be this tangent line and P the point of tangency. Then the line OP is perpendicular to ℓ. Recall that if line ℓ has slope $m \neq 0$, then the lines perpendicular to ℓ have slope $-1/m$. The slope of ℓ is $-3/4$, so OP has slope $4/3$.

If the x-coordinate of P is a then the y-coordinate is $4a/3$. But P is on the circle, so $a^2 + 16a^2/9 = 1$ and therefore $a = 3/5$, and $P = (3/5, 4/5)$. But $3(3/5) + 4(4/5) = 5$, so the maximum of $3x + 4y$ as (x, y) ranges over the disk is 5.

Another Way. Give $3x+4y$ some convenient positive value, say 1. We *minimize* $x^2 + y^2$, that is,

$$x^2 + \frac{(1-3x)^2}{16}.$$

Simplify and complete the square. The minimum value of

$$\frac{\left(5x - \frac{3}{5}\right)^2 + \frac{16}{25}}{16}$$

is $1/25$. So the circle tangent to the line $3x + 4y = 1$ has radius $1/5$. Scale up by a factor of 5: the positive k for which $3x + 4y = k$ is tangent to $x^2 + y^2 = 1$ is 5.

Another Way. The general point on the unit circle has coordinates

$$x = \cos\theta, \qquad y = \sin\theta$$

for some θ between 0 and 2π. It follows that

$$3x + 4y = 5\left(\frac{3}{5}\cos\theta + \frac{4}{5}\sin\theta\right).$$

Let ϕ be the angle between 0 and $\pi/2$ whose sine is $3/5$. Then $\cos\phi = 4/5$, and therefore

$$3x + 4y = 5(\sin\phi\cos\theta + \cos\phi\sin\theta) = 5\sin(\theta + \phi).$$

Choose θ so that $\theta + \phi = \pi/2$. Then $5\sin(\theta + \phi) = 5$, and no larger value is possible.

Another Way. We present an algebraic solution that may look like magic. It uses an important identity that can be checked by expanding both sides:

$$(a^2 + b^2)(x^2 + y^2) = (ax \pm by)^2 + (ay \mp bx)^2.$$

Let $a = 3$ and $b = 4$. We want to make $ax + by$ as large as possible subject to the condition $x^2 + y^2 \leq 1$. From the above identity, we can see that to make $(ax + by)^2$ big, we should scale so as to make $x^2 + y^2 = 1$—that makes the left-hand side as big as possible—and then choose $(ay - bx)^2$ as small as possible. There are x and y such that $x^2 + y^2 = 1$ and $ax - by = 0$, so the maximum value of $(ax + by)^2$ is $a^2 + b^2$.

Comments.

1. The movement as k changes should be experienced kinesthetically. With a picture of the disk and a moving long ruler, we can see and feel the maximum k being reached just as we leave the disk behind, that is, at the point of tangency.

2. In the second solution, the quantity to be maximized and the constraint were in a sense interchanged, and the problem was turned into a minimization problem. This widely useful technique is sometimes called *dualizing* the problem.

3. The identity of the fourth solution is often called *Bachet's Identity,* even though Diophantus used the result more than 1350 years before Bachet. The identity is now sometimes called *Brahmagupta's Identity,* after the seventh-century mathematician who worked with the more general identity

$$(a^2 - Db^2)(x^2 - Dy^2) = (ax \pm Dby)^2 - D(ay \mp bx)^2.$$

   Bachet's Identity can be used to express integers as sums of squares. Let c be a sum of two perfect squares, say $c = a^2 + b^2$, and let $z = x^2 + y^2$. The identity says that cz is a sum of two squares. For example, $401 = 1^2 + 20^2$ and $97 = 4^2 + 9^2$. Using Bachet's Identity, we can express $401 \cdot 97$ as the sum of two squares in two different ways.

4. Here is a seemingly more complicated but closely related problem: maximize $3x + 4y$ given that $x^2 + y^2 \leq 2x + 4y + 6$. By completing the squares we can rewrite the condition as $(x - 1)^2 + (y - 2)^2 \leq 1$. Let $u = x - 1$ and $v = y - 2$. Then $3x + 4y = 3u + 4v + 11$. The maximum value of $3u + 4v$ subject to the condition $u^2 + v^2 \leq 1$ is 5, so the required maximum value is 16.

Problem 5. The number 2^{10} is "nearly" 10^3, in the sense that $2^{10}/10^3$ is close to 1. Show that there are positive integers a and b such that

$$0.998 < \frac{2^a}{10^b} < 1.002.$$

(Logarithms may be useful but are not essential.)

Solution. Let $x_1 = 1.024$. Then $x_1 = 2^{10}/10^3$, and x_1 is reasonably close to 1. We try to find numbers x_2, x_3, x_4, and so on of the form $2^a/10^b$ or $10^a/2^b$, such that x_2, x_3, x_4, and so on are successively closer to 1.

We take powers of x_1 until we get *near* something interesting of the form $2^x/10^y$ or $10^x/2^y$, for example 1.25. Note for instance that

$$(1.024)^9 < 1.25 < (1.024)^{10}.$$

We look at $1.25/(1.024)^9$ and $(1.024)^{10}/1.25$, and pick out the one closer to 1; this happens to be $1.25/(1.024)^9$, which is roughly equal to 1.009742.

Since $1.25 = 10/2^3$, and $1.024 = 2^{10}/10^3$, we conclude that

$$\frac{10/2^3}{2^{90}/10^{27}}$$

is roughly 1.009742. A little manipulation of exponents now tells us that $10^{28}/2^{93}$ is roughly 1.009742. Let $x_2 = 10^{28}/2^{93}$.

Take powers of x_2 until we get *near* something interesting of the form $2^x/10^y$ or $10^x/2^y$. Here x_1 is a reasonable choice; there are plenty of others.

$$(1.009742)^2 \approx 1.019578 \qquad \text{and} \qquad (1.009742)^3 \approx 1.0295115.$$

We look at 1.024/1.019578 and 1.029511/1.024 and pick out the smaller one, which is 1.024/1.019578, approximately 1.0043368. We conclude that

$$\frac{2^{10}/10^3}{10^{56}/2^{186}}$$

is close to 1. But this number is $2^{196}/10^{59}$. Call it x_3.

Take powers of x_3 until we get near x_2. Note for example that x_3^2 is about 1.00869136, pretty close to x_2. In fact, the ratio x_2/x_3^2, which we call x_4 is approximately equal to 1.00104. (The next power of x_3 is quite far from x_2). We have

$$\frac{x_2}{x_3^2} = \frac{10^{28}/2^{93}}{2^{392}/10^{108}} = \frac{10^{136}}{2^{485}}.$$

It is now easy to see that if we let $a = 485$ and $b = 136$, then

$$0.998 < \frac{2^a}{10^b} < 1.002.$$

Comment. We can use this procedure to get numbers of the form $2^a/10^b$ that are arbitrarily close to 1. The basic idea goes as follows. Suppose that we have found a number x_k of the right shape which is close to 1. Now let w_k be (almost) any number say greater than 1 and of the right shape. Take powers of x_k until we get close to w_k, specifically find an exponent d such that

$$x_k^d < w_k < x_k d + 1.$$

It is clear that one of w_k/x_k^d or x_k^{d+1}/w_k is less than or equal to $\sqrt{x_k}$. Call this number x_{k+1}. Then x_{k+1} is of the right shape, and is substantially closer to 1 than x_k was. We could actually choose w_k to be 2 for every k. But in our calculation above, we chose $x_1 = 1.024$, and $w_1 = 1.25$, and for $k \geq 2$ we chose $w_k = x_{k-1}$. This was done in order to make sure that exponents did not rise too fast.

Another Way. We would like to have

$$0.998 < \frac{2^a}{10^b} < 1.002.$$

Take the logarithm of both sides. Here by logarithm we mean logarithm to the base 10. (Logarithms to the base 10 hardly ever occur in mathematics. One sees them mainly in "legacy" formulas in engineering, physics, or chemistry.)

So we would like to have

$$\log(0.998) < a\log 2 - b < log(1.002).$$

The calculator says that $\log(0.998)$ is approximately -0.0008695 and $\log(1.002)$ is approximately 0.0008677. Let $\epsilon = 0.0008$. If we can show that there is a positive integer a such that $a\log 2$ is within ϵ of an integer, then we will know that there exist positive integers a and b with the desired property.

We will show that in fact for *any* positive ϵ, however small, there is a positive integer a such that $a\log 2$ is within ϵ of an integer.

It is enough to show that for any positive integer N, there is a positive integer a such that $a\log 2$ is within $1/N$ of an integer.

First we show that if y is a positive integer, then $y\log 2$ cannot be an integer. Suppose to the contrary that $y\log 2 = x$, where x is a (positive) integer. Then

$$10^{(\log 2)y} = 10^x, \qquad \text{so} \qquad 2^y = 10^x.$$

But this is impossible, since 10^x is divisible by 5, but 2^y is not. We have proved here that $\log 2$ is not a *rational* number. We complete the argument by proving the following useful general result.

Lemma. Let w be a positive irrational number, and let N be a positive integer. Then there exists a positive integer a such that aw is within $1/N$ of an integer.

Proof. For any positive real number x, let $\{x\}$ be the *fractional part* of x, that is, let $\{x\} = x - \lfloor x \rfloor$, where $\lfloor x \rfloor$ denotes the greatest integer which is less than or equal to x. Now look at the $N+1$ numbers $\{w\}$, $\{2w\}$, $\{3w\}$, $\{4w\}$, …, $\{(N+1)w\}$. These numbers are all different, since w is irrational, and they all lie strictly between 0 and 1. It follows that two of these numbers, say $\{cw\}$ and $\{dw\}$, where $1 \leq c < d \leq N+1$, lie at distance less than $1/N$ from each other. Thus the absolute value of $\{(d-c)w\}$ is less than $1/N$, which means that $(d-c)w$ is within $1/N$ of an integer. So we can take $a = d - c$. □

Comment. Note that the proof of the lemma is indirect, and that the only thing we learn about a is that it can be taken to be $\leq N$. For $\epsilon = 0.0008$, we can use $N = 1/0.0008 = 1250$. So we know that there is a positive integer $a \leq 1250$ such that $a \log 2$ is within 0.0008 of an integer. Now that we know this, we could patiently try every a from 1 to 1250, with the assurance that we would find an a that works. But in general that is a very inefficient way to proceed.

There are efficient methods for actually finding an appropriate a, for example *continued fractions.* You may want to search the Internet for information about these. Using continued fractions we find fairly quickly that $\log 2$ is approximately $3/10$, and that substantially better approximations are given by $59/196$ and $146/485$, which gets us to the numbers we sort of bumped into in the first solution. I think this is not entirely a coincidence. Might be worth looking into.