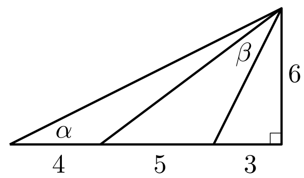
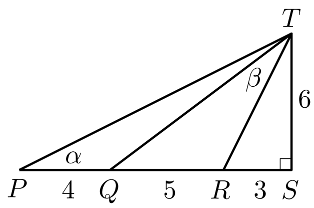
Solutions to March 2009 Problems

Problem 1. In the right-angled triangle below, dimensions are as shown. Prove that $\alpha = \beta$.



Solution. Since we cannot simply point, it is useful to label the diagram.



Using the Pythagorean Theorem, we find that $PT = 6\sqrt{5}$, $QT = 10$, and $RT = 3\sqrt{5}$. Note that

$$\frac{TP}{QT} = \frac{RT}{RQ} = \frac{PR}{TR}.$$

Hence $\triangle PRT$ is similar to $\triangle TRQ$. It follows immediately that $\alpha = \beta$.

Another Way. As in the first solution, use the Pythagorean Theorem to compute PT, QT, and RT.

Using $\triangle PST$, we conclude that $\cos(\alpha) = 12/(6\sqrt{5}) = 2/\sqrt{5}$. Using the Cosine Law on $\triangle QTR$, we find that

$$5^2 = 10^2 + (3\sqrt{5})^2 - 2(10)(3\sqrt{5})\cos(\beta).$$

Simplify, and solve for $\cos(\beta)$. We obtain $\cos(\beta) = 2/\sqrt{5}$. Since $\cos(\alpha) = \cos(\beta)$, we conclude that $\alpha = \beta$.

Another Way. Note that $\tan(\alpha) = 6/12$, and $\tan(\angle RTS) = 3/6$. Thus $\angle RTS = \alpha$.

It follows that $\angle STQ = \alpha + \beta$. Recall (or prove using the addition laws for sine and cosine) that in general

$$\tan(\alpha + \beta) = \frac{\tan(\alpha) + \tan(\beta)}{1 - \tan(\alpha)\tan(\beta)}.$$

But $\tan(\alpha+\beta) = 8/6 = 4/3$ and $\tan(\alpha) = 1/2$. Thus

$$\frac{4}{3} = \frac{\frac{1}{2} + \tan(\beta)}{1 - \frac{1}{2}\tan(\beta)}.$$

Solve for $\tan(\beta)$. We find that $\tan(\beta) = 1/2$, and therefore $\beta = \alpha$.

Problem 2. Let $f(x) = ax^2+bx+c$, where a, b, and c are real numbers. Define $f'(x)$ by $f'(x) = 2ax + b$. Show that if $|f(x)| \leq 1$ on the interval $[-1, 1]$, then $|f'(x)| \leq 4$ on the same interval. No calculus please!

Solution. We treat separately the simple case where $a = 0$. Since all we care about is absolute values, we may without loss of generality we may assume that b is positive. Since $|f(x)| \leq 1$ on our interval, $|f(1)| \leq 1$ and $|f(-1)| \leq 1$. Thus $b + c \leq 1$ and $-b + c \geq -1$, or equivalently $b - c \leq 1$. From $b + c \leq 1$ and $b - c \leq 1$, we conclude by adding that $2b \leq 2$, and since b is positive, it follows that $|b| \leq 1$. But $f'(x) = b$ for all x, so certainly $|f'(x)| \leq 4$.

Now we deal with the case $a \neq 0$. Since we are working with absolute values, we may without loss of generality assume that $a > 0$. (To put it more geometrically, if $a < 0$ we reflect the curve with equation $y = ax^2 + bx + c$ in the x-axis to obtain an upward facing parabola.)

Suppose that the vertex of the parabola $y = ax^2 + bx + c$ has positive x-coordinate, or equivalently that b is negative. Then reflect the parabola in the y-axis. The largest absolute value of $ax^2 + bx + c$, and of $2ax + b$, remain unchanged, but now the vertex of the parabola has negative x-coordinate. So without loss of generality we may assume that $b \geq 0$.

Since $|f(1)| \leq 1$, we conclude that $a + b + c \leq 1$, so $c \leq 1 - a - b$. But $|f(0)| \leq 1$, and therefore $-1 \leq c$.

It follows that $-1 \leq 1 - a - b$, or equivalently that $a + b \leq 2$. Since a and b are non-negative, it follows that $2a + 2b \leq 4$. Thus $2a + b \leq 4$, with equality only if $b = 0$ (and therefore $a = 2$).

The graph of $y = f'(x)$ is a straight line with positive slope. So to show that $|f'(x)| \leq 4$ on our interval, it is enough to show that $2a + b \leq 4$ and $-2a+b \geq -4$. To show that $-2a+b \geq -4$, it is enough to show that $2a-b \leq 4$. This follows easily from the fact that $2a + b \leq 4$, since b is non-negative. We conclude that if $|f(x)| \leq 1$ on our interval, then $|f'(x)| \leq 4$ on our interval.

Comment. We have proved a little more, since we showed that if $a > 0$ we can only have $|f'(x)| = 4$ if $b = 0$ Thus the only examples where the maximum is reached are given by $f(x) = 2x^2 - 1$ and $f(x) = -2x^2 + 1$.

The maximality property of $2x^2-1$ turns out to be connected to the fact that $\cos(2\theta) = \cos^2(\theta) - 1$. And there is an important related class of polynomials, the *Chebyshev polynomials*, for which related results can be proved. Chebyshev polynomials are widely useful.

Problem 3. Show that $n^{n-1}-1$ is divisible by $(n-1)^2$ for every positive integer n.

Solution. The result is easy to verify for $n = 1$ amd $n = 2$. For $n > 2$, let $P(x)$ be the polynomial given by $P(x) = x^{n-1} - 1$. Then

$$P(x) = (x - 1)(x^{n-2} + x^{n-3} + \cdots + 1).$$

From the above factorization, it is clear that $n - 1$ divides $P(n)$. To show that $(n - 1)^2$ divides $P(n)$, we need to show that $n - 1$ divides

$$n^{n-2} + n^{n-3} + \cdots + 1.$$

The above sum has $n - 1$ terms. Subtract 1 from each term, and add $n - 1$ to the end to compensate. We get

$$(n^{n-2} - 1) + (n^{n-3} - 1) + \cdots + (n - 1) + (1 - 1) + (n - 1).$$

But $n - 1$ always divides $n^k - 1$, so each term above is divisible by $n - 1$. This completes the proof.

Problem 4. How many ordered triples (x, y, z) of integers are there such that $|x| + |y| + |z| \le 1000$?

Solution. It is no harder to count the triples with $|x| + |y| + |z| \le n$. And solutions where some of the variables are 0 are a little different, so we take care of them separately.

There is obviously only 1 ordered triple with $x = y = z = 0$. Next we count the ordered triples with exactly two entries equal to 0. *Which* entries are 0 can be decided in 3 ways. For each such way, the non-zero entry can be selected in $2n$ ways, for a total of $6n$. For reasons that will become clear later, we write instead $6\binom{n}{1}$.

Next we count the ordered triples with exactly 1 entry equal to 0. *Which* entry is 0 can be decided in 3 ways. For each such way, we find the number of ways the non-zero entries can be chosen. So we need to find the number of ordered pairs (u, v) with u and v non-zero and $|u| + |v| \le n$. It is easier to find the number of such ordered pairs with u and v positive, and then to multiply by 4 to deal with the fact that all combinations of signs are allowed.

Here is a neat trick for doing the counting. We want to find the number of triples (u, v, d) with u, v and d positive and $u + v + d = n + 1$. Imagine a long shelf of books with $n + 1$ books. Then there are n "gaps" between books. Choose 2 of these gaps to put dividers into. Let u be the number of books up to the first divider, v the number of books between the two dividers, and finally d the number of books after the second divider. There are exactly as many ways to find positive integers u, v, d with $u + v + d = n + 1$ as there are of placing the dividers. Thus there are $\binom{n}{2}$ pairs (u, v) of positive integers with $u + v \le n$.

We conclude that the number of ordered triples (x, y, z) with exactly one entry equal to 0 and $|x| + |y| + |z| \le n$ is $12\binom{n}{2}$.

Finally, we count the number of ordered triples with no entry equal to 0. It is easier to count the number of ordered triples with all entries positive, and then multiply by 8 to account for the possibility of changing signs. Using the same trick as before, we want to count the number of ordered quadruples (x, y, z, d) of positive integers with $x+y+z+d = n+1$. This is just the number of ways of putting 3 dividers into the n "gaps" between $n+1$ books on a shelf, that is, $\binom{n}{3}$. So the number of ordered triples with no entry equal to 0 is $8\binom{n}{3}$.

Add up. We get a total of

$$1 + 6\binom{n}{1} + 12\binom{n}{2} + 8\binom{n}{3}.$$

Finally, in order to compare this with counts obtained in other ways, set $n = 1000$ and calculate. We get 1335336001.

Another Way. For any integer k, let S_k be the number of solutions of the equation $|x| + |y| = k$. For a fixed number k, with $0 \le k \le n$, the number of solutions of $|x| + |y| + |z| \le n$ is $[2(n-k)+1]S_k$, since for any choice of x, y with $|x|+|y| = k$, the number z can take on all values from $-(2n-k)$ to $2n-k$.

We now find an explicit expression for S_k. It is clear that $S_0 = 1$ and $S_1 = 4$. Now take $k \ge 2$. The solutions of $|x| + |y| = k$ are $(\pm k, 0)$, $(0, \pm k)$, together with the solutions with neither x nor y equal to 0. If $|x|$ and $|y|$ are positive, $|x| = 1$, $|y| = k-1$, or $|x| = 2$, $|y| = k-2$, and so on, altogether $4(k-1)$ possibilities, for a total of $4k$.

Thus the total number of solutions of $|x| + |y| + |z| \le n$ is

$$2n + 1 + \sum_{k=1}^{n} [2(n-k)+1](4k).$$

The above sum is equal to

$$2n + 1 + (2n+1)(4)\sum_{k=1}^{n} k - 8\sum_{k=1}^{n} k^2.$$

Using standard formulas, we find that the sum is

$$2n + 1 + (2n+1)(4)\frac{n(n+1)}{2} - 8\frac{n(n+1)(2n+1)}{6}.$$

We can evaluate the above expression for $n = 1000$, or simplify a bit and then evaluate. The numerical answer is 1335336001.