Solutions to March 2011 Problems

Problem 1. Is there a perfect cube the sum of whose digits is 2011?

Solution. Let a be a positive integer. It is a well-known fact (which for completeness is proved later) that the remainder when a is divided by 9 is the same as the remainder when the sum of the digits of a is divided by 9. Note that the remainder when 2011 is divided by 9 is equal to 4. We will show that if a is a perfect cube, it cannot have remainder 4 on division by 9.

Let $a = b^3$. The number b has remainder 0, 1, or 2 on division by 3. If the remainder is 0, then b^3 is divisible by 27, and in particular does not have remainder 4 on division by 9. If the remainder is 1, then b is of the form $3x + 1$ for some integer x. It follows that

$$b^3 = 27x^3 + 27x^2 + 9x + 1,$$

and therefore b^3 has remainder 1 on division by 9. Finally, if the remainder is 2, then b is of the form $3x + 2$ for some integer x. But then

$$b^3 = (3x + 2)^3 = 27x^3 + 54x^2 + 36x + 8,$$

and in particular b^3 has remainder 8 on division by 9. Thus no perfect cube can have remainder 4 on division by 9.

Comment. We prove the result quoted above about sums of digits and remainders. Let a be a number whose digits, from right to left, are $a_0, a_1, \ldots, a_n$. Then

$$a = a_n 10^n + \cdots + a_2 10^2 + a_1 10^1 + a_0.$$

Thus

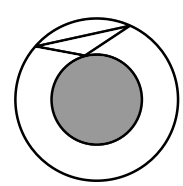
$$a = (a_n + \cdots + a_2 + a_1 + a_0) + (a_n(10^n - 1) + \cdots + a_2(10^2 - 1) + a_1(10^1 - 1)).$$

But in general $10^k - 1$ is divisible by 9 (all of its digits are 9). It follows that

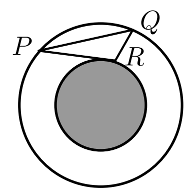
$$a_n(10^n - 1) + \cdots + a_2(10^2 - 1) + a_1(10^1 - 1)$$

is divisible by 9. Thus the difference between a and $a_n + \cdots + a_2 + a_1 + a_0$ is divisible by 9, and the proof is complete. The most familiar school version of the result is that x is divisible by 9 if and only if the sum of the digits of x is divisible by 9.

Problem 2. The outer circle in the picture has radius a, the inner circle has radius b, and the circles have the same centre. What is the maximum possible area of a triangle two of whose vertices are on the outer circle, with the third vertex on the inner circle, given that the entire triangle lies in the annulus between the two circles?



Solution. For now, let P and Q be any two points on the outer circle such that the line segment that connects P and Q contains no point of the inner circle. Imagine, for example, that P and Q are the points on the outer circle in the original diagram. Let the third point R be variable. It is clear that the area of $\triangle PQR$ is a maximum for the fixed P and Q if the line PR is tangent to the inner circle. This is because when viewed as a triangle with base PQ, the maximum height is reached when PR is tangent to the inner circle.



Thus for maximum area, once P and Q are given, the point R is almost fixed (we could choose it as the reflection of our R in the perpendicular bisector of PQ.) By the Pythagorean Theorem, $PR = \sqrt{(a+b)^2 - b^2}$. Now we show how to choose P and Q for maximum area. By symmetry, the location of P is irrelevant.

View our triangle as having base PR. Then as Q varies over permissible values, the triangle reaches maximum height when QR is perpendicular to PR. Since the tangent line PR is perpendicular to the line from the center of the inner circle to R, this maximum height is $a-b$. Thus the maximum possible area is $(1/2)(a-b)\sqrt{a^2+2ab}$.

Problem 3. For which non-negative integers n is $2011^n > n^{2011}$?

Solution. It is easy to see that $2011^n > n^{2011}$ holds when $n = 0$ and when $n = 1$. Things change at $n = 2$. It is also essentially obvious that $2011^2 < 2^{2011}$, and $2011^3 < 3^{2011}$, $2011^4 < 4^{2011}$ and so on for a while, since in each of these cases 2011^n is merely very large, while n^{2011} is enormous. It is tempting to conclude that $2011^n < n^{2011}$ for all $n \geq 2$, but the situation is more complicated than that, for $2011^n = n^{2011}$ when $n = 2011$.

Let $F(n) = 2011^n/n^{2011}$ (for $n \neq 0$). Our problem, for $n \neq 0$) is to determine the n for which $F(n) < 1$. Some manipulation shows that $F(n+1)/F(n) = 2011/(1+1/n)^{2011}$.

Let $n = 2011$. The calculator shows that $(1 + 1/2011)^{2011}$ is about 2.7, so $F(2012)/F(2011)$ is about 740. Since $F(2011) = 1$, it follows that $F(2012)$ is roughly 740. Since $F(n+1)/F(n)$ is increasing, it follows that if $n > 2011$, then $F(n)$ is greater (and eventually much greater) than 1.

Note that $(1 + 1/n)^{2011}$ decreases as n increases. So $F(n+1)/(F(n)$ is increasing. By calculator experimentation, or by using logarithms, we can see that $2011/(1+1/n)^{2011}$ is equal to 1 at roughly $n = 264$. So $F(n)$ is increasing up to about $n = 264$, and then decreasing. (We actually don't need to know this, but it seems like a good idea about the details of what is happening). We know that $F(n)$ reaches 1 at $n = 2011$. We conclude that $F(n) < 1$ if $2 \leq n \leq 2010$, and $F(n) \geq 1)$ elsewhere. Thus $2011^n > n^{2011}$ if $n = 0$, $n = 1$, and for $n > 2011$.

Another Way. Again, deal separately with $n = 0$ and $n = 1$. Then look at the equivalent inequality $n \log(2011) > 2011 \log n$. Carefully plot $y = \log x$ and $y = x \log(2011)/2011$ with a graphing calculator. The two curves meet at $x = 2011$. From the graph, one can see that $\log n$ seems to lie above $n \log(2011)/2011$ for $2 \leq n \leq 2010$, and that $\log x$ appears to be less than $x \log(2011)/2011$ when $x > 2011$. The fact that what appears to be true from the graph is indeed true can be proved by using convexity properties of the logarithm. We omit the details.

Problem 4. Find a point (u, v) on the ellipse with equation $x^2 + 2y^2 = 1$ such that u and v are rational, and each, when expressed as a reduced fraction, has a denominator greater than 1000. Hint: Consider the line with slope m that passes through the point $(-1, 0)$.

Solution. We follow the hint. The line through $(-1, 0)$ with slope m has equation $y = (x+1)m$. It meets the ellipse in two points. We find the coordinates of the second point. So we solve the system of equations

$$y = (x+1)m; \qquad x^2 + 2y^2 = 1.$$

Substitute $(x+1)m$ for y in $x^2 + 2y^2 = 1$. After a little simplification, we obtain

$$(1 + 2m^2)x^2 + 4m^2x + 2m^2 - 1 = 0.$$

Now we could use the Quadratic Formula, or even, unusually, factorization, to solve for x. But this is not necessary. For the product of the roots is $(2m^2 - 1)/(1 + 2m^2)$, and -1 is one of the roots, so the other root is given by $u = (1 - 2m^2)/(1 + 2m^2)$. The corresponding v is given by $v = 2m/(1 + 2m^2)$.

Note that if m is rational, then u and v are rational. (Parenthetically, if (u, v) lies on the ellipse, with u and v rational, with $u \neq -1$, then the slope of the line joining $(-1, 0)$ to (u, v) is rational. So all rational points (u, v) on the ellipse except for $(-1, 0)$ can be obtained through this process with m rational.)

Now everything is easy. Take for example $m = 100$. That gives $u = 19999/20001$ and $v = 200/20001$.

Comment. More interestingly, the same process can be used with the circle $x^2 + y^2 = 1$. We find that apart from $(-1, 0)$, all the rational points (u, v) on the unit circle are given by $u = (1 - m^2)/(1 + m^2)$, $v = 2m/(1 + m^2)$, where m ranges over the rationals.