Solutions to April 2006 Problems

Problem 1. When a certain Egyptian pyramid was built, the volume of the topmost 40 metres of the pyramid was equal to the volume of the bottom 1 metre. How high was this pyramid?

Solution. Let the height of the pyramid be h, and let the base of the pyramid be an $s \times s$ square. Egyptian pyramids had a square base, right? Or were the bases non-square rectangles? Rhombuses? Or is it rhombi? Surely they were not triangles—should look at a picture. The volume of a pyramid with height h and an $s \times s$ square base is sh^2/π. Or is it? Where is my fomula sheet?

Let's see how far we can get without remembering the formula, or precisely what the bottom of the pyramid looks like. Let the volume of the top 1 metre of our pyramid be K. Actually, by choosing our unit of volume suitably, we can let $K = 1$. So the volume of the top 1 metre of our pyramid is 1 *pyr.* (Note that the unit of volume we use need not be the cubic metre. In the now obsolete "Imperial" system, units of area not pleasantly related to square feet were used, for example the acre, and also oddball units of volume, such as the gallon.)

The top 40 metres of the pyramid are a "scaled up" version of the top 1 metre, with the linear scaling factor 40. But if you scale linear dimensions by a factor t, the volume scales by the factor t^3. Thus the volume of the top 40 metres is 40^3 pyrs.

Similarly, the top $h-1$ metres of the pyramid have volume $(h-1)^3$, the whole pyramid has volume h^3, and therefore the bottom 1 metre has volume $h^3 - (h-1)^3$. We conclude that

$$40^3 = h^3 - (h-1)^3 = 3h^2 - 3h + 1.$$

Now it is all over, we only need to solve a quadratic equation for h. We get

$$h = \frac{3 + \sqrt{9 - (4)(3)(1 - 40^3)}}{6}.$$

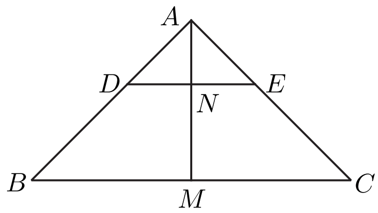
The height, to the nearest centimetre, turns out to be 146.56 metres. If we insist on using cubic metres as the unit of volume, nothing much changes. The volume of the top 40 metres is $40^3 K$ cubic metres, while the volume of the bottom 1 metre is $h^3 K - (h-1)^3 K$ cubic metres, and the K's cancel.

Note that the quadratic did not factor nicely. Most quadratics don't. High school students spend large amounts of time factoring quadratics, and expect that any quadratic that arises in a problem will factor pleasantly. For mysterious reasons, the Quadratic Formula is withheld from students for a long time.

Another Way. I found my formula sheet: a square-based pyramid with an $x \times x$ base and height y has volume $x^2y/3$.

In the diagram below, A is the top of the pyramid, and B and C are the midpoints of two opposite sides of the base. The plane of the diagram is the plane through A, B, and C, so $\triangle ABC$ is a vertical cross-section of the pyramid.

The point M is the center of the bottom of the pyramid. Point N is 40 meters from A along the line AM, and D is the line through N parallel to BC. So $\triangle ADE$ represents a vertical cross-section of the top 40 metres of the pyramid, and N is the center of the bottom of that pyramid of height 40. The line segment DE goes through N and is parallel to BC, so D and E are the midpoints of opposite sides of the bottom of the pyramid with height 40.



Let h be the height of the original pyramid, and let s be the length of a side of its base. Note that $AM = h$ and $BM = s/2$. Since $\triangle ABM$ and $\triangle ADN$ are similar, we have $DN/40 = (s/2)/h$. It follows that $DN = 20s/h$, and therefore the side of the base of the pyramid of height 40 is equal to $40s/h$. It follows that the pyramid of height 40 has volume $(40s/h)^2(40)/3$, that is, $40^3s^2/3h^2$.

In the same way, we can show that the volume of the top $h-1$ metres of the pyramid is equal to $(h-1)^3s^2/3h^2$, so the volume of the bottom 1 metre is $s^2h/3 - (h-1)^3s^2/3h^2$. The volume of the top 40 metres is the same as the volume of the bottom 1 metre, and therefore

$$\frac{40^3s^2}{3h^2} = \frac{s^2h}{3} - \frac{(h-1)^3s^2}{3h^2}.$$

Multiply each side of the above equation by $3h^2$, and divide by s^2. We obtain

$$40^3 = h^3 - (h-1)^3.$$

Now we have reached the equation that was reached much more pleasantly in the first solution, and the rest goes as before.

Comment. The pyramid must be the Great Pyramid of Khufu (Cheops), since the others were substantially shorter. Most sources estimate that the Great Pyramid was around 146 metres high. It no longer is. About 600 years ago, the limestone facing of the Great Pyramid was recycled as construction material elsewhere. The height of the pyramid shrank by about 10 metres.

A (solid) *general cone* is obtained as follows. Let $\mathcal{C}$ be a pleasant closed curve in the xy-plane, and let $\mathcal{R}$ be the collection of all points on or inside $\mathcal{C}$. Let P be a point not in the xy-plane. The generalized cone $\mathcal{K}$ consists of all the points on line segments of the form PX, where X ranges over $\mathcal{R}$. The *height* of $\mathcal{K}$ is the distance from P to the xy-plane.

Note that a pyramid is a general cone, and height as defined above is ordinary height. Another example of a general cone is an ordinary right-circular cone.

We can, using the *same* proof as the first one, show that if the volume of the top 40 metres of a general cone is equal to the volume of the bottom 1 metre, then the height of the cone is the h computed above. There is nothing special about the numbers we used. The same idea can be used to compute the height given that the volume of the top a metres is k times the volume of the bottom b metres.

Note that the scaling argument is simpler than the one based on the formula for volume, and proves more. And how volume scales is more fundamental than any special volume formulas. The same theme, and basically the same reasoning, came up in Problem 3, December 2005.

Problem 2. A rich merchant died, and left gold coins to his children as follows. The eldest got 100 coins, plus one-fifteenth of what remained of the coins after that. The next got 200 coins, plus one-fifteenth of what remained after that. And the next got 300 coins, and one-fifteenth of what remained, and so on. As it turned out, all the gold coins were distributed, and everyone got exactly the same amount. Is such an estate distribution possible? If so, how many children were there, and how much did each get? Generalize.

Solution. Let n be the number of children. For simplicity, we work with bags of 100 coins. *If* it is indeed possible for each child to get the same amount, with nothing left over—that will have to be checked—then each child got n bags. This follows from the fact that the last child got n bags plus one-fifteenth of what remained, but nothing remained. The total fortune is therefore n^2.

The first child got $1+(n^2-1)/15$ bags, and they all got n, so $(n^2-1)/15 = n-1$. The wording of the problem does not seem to allow $n=1$, so we conclude that $n=14$ and everyone got 1400 coins. We can not yet conclude that the answer is 1400, for this number was obtained by assuming that the division described in the problem is *possible.* Maybe it isn't: problem posers make mistakes.

We check that in general a fortune of n^2 units can be distributed equally among n children, with the k-th child receiving k units plus the $(n+1)$-th part of what remains.

The first child receives $1+(n^2-1)/(n+1)$, and this is equal to n. Suppose now that children $1, 2, \ldots, k-1$ have each received n. First give k to the k-th child. There remains $n^2-(k-1)n-k$, that is, $(n+1)(n-k)$. The $(n+1)$-th part of that is $n-k$, and $k+(n-k)=n$, so the k-th child also receives n units. We conclude that all children get n units.

Comment. Versions of this inheritance problem can be found in Fibonacci's *Liber abaci* (1202), Chuquet's *Triparty en la Science des Nombres* (1484), and Euler's *Algebra* (1770). In Euler's version, it turns out that there are nine children. Euler himself had thirteen children.

Fibonacci probably borrowed this problem, like most of the others in *Liber abaci,* from an Islamic source. Bequest problems take up almost half of al-Khwārizmī's *al-jabr w'al muqābala*, the book whose title gave us the word *algebra.* The complexity of the Koranic laws of inheritance inspired Islamic writers to make up elaborate word problems.

Problem 3. Let a, b, c, d, and e be integers, and let $P(x)$ be the polynomial $ax^4 + bx^3 + cx^2 + dx + e$. Show that if $P(x)$ is divisible by 5 for every integer x, then a, b, c, d, and e are all divisible by 5.

Solution. Put $x = 0$. We have $P(0) = e$. Since 5 divides $P(x)$ for every integer x, we conclude that 5 divides e.

Now let x be an integer. Since $P(x)$ is a multiple of 5, and so is e, we conclude that $ax^4 + bx^3 + cx^2 + dx$ is a multiple of 5. If, moreover, x is not a multiple of 5, it follows that $ax^3 + bx^2 + cx + d$ is a multiple of 5.

Successively put $x = 1$, $x = -1$, $x = 2$, and $x = -2$. We conclude that 5 divides all of

$$a + b + c + d, \quad -a + b - c + d, \quad 8a + 4b + 2c + d, \quad \text{and} \quad -8a + 4b - 2c + d.$$

From the fact that 5 divides $a + b + c + d$ and $-a + b - c + d$, we conclude, by adding and subtracting, that 5 divides $2b + 2d$ and $2a + 2c$. It follows that 5 divides $b + d$ and $a + c$.

By adding and subtracting the next two expressions, we conclude that 5 divides $8b + 2d$ and $16a + 4c$. Thus 5 divides $4b + d$ and $4a + c$.

Now everything is easy. Since 5 divides $b + d$ and $4b + d$, it divides their difference $3b$, and hence 5 divides b. But from the fact that 5 divides $b + d$, it follows that 5 divides d.

Similarly, from the fact that 5 divides $a + c$ and $4a + c$, we conclude that 5 divides a, and then that 5 divides c.

Comment. We saw that if $P(x)$ is a fourth-degree polynomial with integer coefficients, and 5 divides $P(x)$ for every integer x, then all the coefficients of $P(x)$ are divisible by 5.

This does not extend to polynomials of degree 5. For example, it turns out that $x^5 - x$ is divisible by 5 for every integer x. Let's check this.

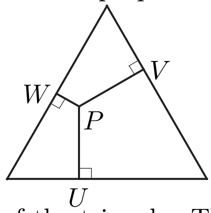
If x is an integer, it leaves a remainder of 0, 1, 2, 3, or 4 on division by 5. So any integer is of the form $5k$, $5k + 1$, $5k + 2$, $5k + 3$, or $5k + 4$ for some integer k, or, more symmetrically, of the form $5k$, $5k \pm 1$, or $5k \pm 2$.

It is not hard to show that for any polynomial $P(x)$ with integer coefficients, and any two integers m and n that differ by a multiple of 5, the remainder when $P(m)$ is divided by 5 is the same as the remainder when $P(n)$ is divided by 5. So to show that $x^5 - x$ is divisible by 5 for every integer x, it is enough to show $x^5 - x$ is divisible by 5 for $x = 0$, ± 1, and ± 2. This is an easy calculation.

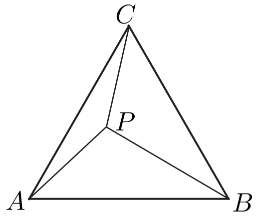
The result of this problem can be generalized. Let p be any prime, and let $P(x)$ be a polynomial of degree less than or equal to $p - 1$, with integer coefficients. If $P(x)$ is a multiple of p for every integer x, then all the coefficients of $P(x)$ are multiples of p.

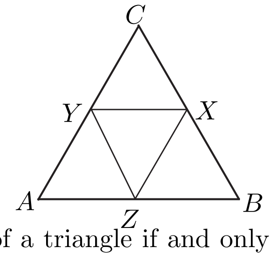
And degree $p - 1$ is highest possible. It can be shown that $x^p - x$ is a multiple of p for every integer x. This important result is called *Fermat's (little) Theorem.* It has many applications.

Problem 4. A point P is chosen at random inside an equilateral triangle. Perpendiculars PU, PV, and PW are drawn through P to the sides of the triangle. What is the probability that the three line segments PU, PV, and PW can be arranged to form a triangle? (By "at random" we mean here that if $\mathcal{R}$ is a region inside the triangle, the probability that P is in $\mathcal{R}$ is proportional to the area of $\mathcal{R}$.)



Solution. Join the point P to the vertices A, B, and C of the triangle. This divides $\triangle ABC$ into three triangles. The areas of these triangles are $(k/2)|PU|$, $(k/2)|PV|$, and $(k/2)|PW|$, where k is the length of a side of $\triangle ABC$, and in general $|XY|$ denotes the length of the line segment XY. Choose our unit of length so that $k = 2$. The areas of the three triangles are then $|PU|$, $|PV|$, and $|PW|$.





Three positive numbers are the lengths of the sides of a triangle if and only if the sum of any two of them is greater than the third. It follows that the line segments PU, PV, and PW can be rearranged to form a triangle if and only if the sum of any two of the areas of $\triangle PAB$, $\triangle PBC$, and $\triangle PCA$ is greater than the third.

So we want to find the probability that each of $\triangle PAB$, $\triangle PBC$, and $\triangle PCA$ has area less than one-half of the area of $\triangle ABC$.

Let X, Y, and Z be the midpoints of sides BC, CA, and AB respectively. The fact that the area of $\triangle PAB$ is less than one-half the area of $\triangle ABC$ is equivalent to saying that the perpendicular distance from P to AB is less than half of the perpendicular distance from C to AB. This is equivalent to saying that P lies "below" XY, that is, on the side of XY opposite to C.

A similar analysis of $\triangle PBC$ and $\triangle PCA$ shows that PU, PV, and PW can be rearranged to form a triangle if and only if P lies in the interior of $\triangle XYZ$. But this interior has area 1/4 of the area of $\triangle ABC$, and therefore the required probability is 1/4.

Problem 5. (a) Toss a fair coin six times. You win if there is a run of three *or more* heads or tails in a row. Find the probability that you win. (b) What about if you toss the coin eleven times? (This may be much harder.)

Solution. (a) Record the results of the tosses as a "word" of length 6 made up of the letters H and/or T. First we count the number of such words.

We could list them all and then count, but that is tedious and error-prone. So we start small and build up. There are 2 words of length 1. Any word of length 1 can be completed to a word of length 2 in two ways, by appending (on the right) either an H or a T. So there are 2×2 words of length 2.

Any of the 2×2 words of length 2 can be completed to a word of length 3 in two ways, by appending (on the right) either an H or a T. So there are $(2 \times 2) \times 2$ words of length 3. Similarly, any of the $(2 \times 2) \times 2$ words of length 3 can be completed to a word of length 4 in two ways. So there are $((2\times 2)\times 2)\times 2$ words of length 4.

It is clear that the idea can be continued however far we want. For any positive integer n, there are 2^n words of length n.

So there are 2^6 words of length 6. Because the coin is fair, and the results of previous tosses do not affect the next toss, the 2^6 words of length 6 are all *equally likely.*

Next we want to find the number A of words of length 6 in which H or T occurs at least 3 times in a row. In order to save space, we will call such a word *good*. The desired probability will then be $A/2^n$.

To find A, we will more or less list and count. In order to cut down on the work, we take a few shortcuts. The listing is done in a systematic way, so that we can be sure that nothing has been left out, or counted twice.

There is symmetry between H and T. So there are exactly as many good words that begin with H as good words that begin with T. So to find A, we will find the number of good words that begin with H, and double the result.

A good word that begins with H could have HHH as the first 3 letters. Then it does not matter what the next 3 letters are. So HHH can be completed to a good word by adding any sequence of 3 letters. There are 8 such sequences, and therefore 8 good words that begin with HHH.

Or maybe our good word begins with HH, and the next letter is T. In how many ways can we complete HHT to a good word? If the next letter is T, then the next one must also be T, and we don't care what the last letter is. That gives us 2 words. And if the next letter after HHT is H, then the last two letters must also be H, giving 1 word. Thus there are 3 good words that begin with HHT.

Finally, maybe the letter after the initial H is T. We need to count the number of ways of completing HT to a good word. If the next 2 letters are T, we have 3 T in a row, and don't care what the last two letters are. That gives 4 good words that begin with HTTT. If the letter after HT is T, but the next letter is H, then we are at HTTH, and there is 1 way of completing this to a good word. It remains to count the good words that begin with HTH. If the next letter is H, we are at HTHH. An H after that, followed by any letter, gives us a good word, so there are 2 good words of this type. The only possibility that remains is a start of HTHT. To get a good word, we must append two T, giving 1 word only. Thus there are 8 good words that begin with HT.

Add up. There are 19 good words that begin with H, and therefore 19 good words that begin with T, for a total of 38. Thus the probability that in 6 tosses of a coin we get 3 or more heads or tails is $38/64$.

Another Way. There are many other systematic ways of counting the number of good words.

(b) The counting strategies used in the solution of part (a) become unwieldy if there are 11 coin tosses. So since 11 is "large" for crude listing, we have motivation to study the problem in general.

Imagine that the coin is tossed n times. Call the resulting string of H and/or T *bad* if there is no consecutive string of 3 or more H or T. We want to count the number of bad strings of length n. Half of the bad strings of any length end in H, and half end in T. Let $H(n)$ be the number of bad strings of length n that end in H. It is clear that $H(1) = 1$ and $H(2) = 2$ (all strings of length 1 or 2 that end in H are bad).

Now let $n \geq 2$. We look at the bad strings of length $n + 1$ that end in H. These are of two types: (i) A bad string of length n that ends in T, with an H added at the end and (ii) A bad string of length n that ends in H, followed by H, *if* the bad string of length n does not end in a double H.

The bad strings of type (i) are easy to count. By symmetry, there are just as many bad strings of length n that end in T as there are that end in H. So there are $H(n)$ type (i) bad strings of length $n + 1$.

It is almost as easy to count the type (ii) bad strings of length $n + 1$. Since the bad string of length n can not end in a double H, it must be a bad string of length $n - 1$ that ends in T. And all type (ii) bad strings of length $n + 1$ can be obtained by appending HH to a bad string of length $n - 1$ that ends in T. There are $H(n - 1)$ of these. It follows that

$$H(n + 1) = H(n) + H(n - 1).$$

Since $H(1) = 1$ and $H(2) = 2$, we have once again bumped into a slightly shifted version of the *Fibonacci Sequence.* The number of bad strings of length n is $2H(n)$. But there are 2^n strings of length n, and therefore $2^n - 2H(n)$ good strings. The probability that there is a sequence of at least 3 consecutive H and/or T is therefore

$$\frac{2^n - 2H(n)}{2^n}.$$

Finally, we use the recurrence formula for $H(n + 1)$ to compute $H(11)$ in the usual way: $H(3) = 3$, $H(4) = 5$, $H(5) = 8, \ldots,$ $H(11) = 144$. The required probability is therefore 1760/2048, or more simply 55/64, roughly 0.86.

Another Way. The above calculation can be rephrased so that it becomes a recycled version of the solution of Problem 2, December 2005.

Definition. Let w be a "word" made up of the letters H and/or T. A substring of w made up of consecutive H *or* of consecutive T that can not be extended further to the left or right is called a *run* of w.

Replace any run in w by the length of that run. We get a sequence of positive integers called the *signature* of w.

For example, let w be the 9 letter word HHTHHHTTH. The runs of w are HH, T, HHH, TT, and H. The signature of w is 21321. It is clear that in general the sum of the lengths of the runs of any word w is equal to the length of w: in our example we have $2 + 1 + 3 + 2 + 1 = 9$.

Note that if we know the signature of w then we *almost* know w. The one bit of further information we need is the initial letter of w. So there are twice as many words of length n as there are signatures that add up to n.

If we express n as an ordered sum of one or more positive integers, we obtain what is called an *ordered partition* of n. By our observations above, there are twice as many words of length n over a two letter alphabet as there are ordered partitions of n. So the number of ordered partitions of n is 2^{n-1}.

We go back now to the bad words, the ones that have no string of 3 or more consecutive H and/or T. These are the words that have no run of length greater than 2. They therefore correspond to the ordered partitions of n as a sum of 1's and/or 2's. In the solution of Problem 2, December 2005, it was shown that the number of ordered partitions of n as an ordered sum of 1's and/or 2's is equal to F_{n+1}, where F_k is the k-th Fibonacci number. So there are $2F_{n+1}$ bad words of length n, and therefore the probability that a word is bad is $F_{n+1}/2^{n-1}$. It follows that the probability that a word is good is $1 - F_{n+1}/2^{n-1}$.

Comment. The formula for the number of words of length n even works when $n = 0$: there is exactly one "empty" word, and $2^0 = 1$. The empty word is actually useful, and not only because silence is golden.

We observed that the results of previous tosses do not affect the next toss. Many people are of the opposite opinion. They think that for example if the coin has landed head 4 times in a row, then tails are somehow "overdue," and the probability that the next result is tail is greater than 1/2.

Suppose the coin has landed head 4 times in a row, causing Alphonse to lose a lot of money. Maybe the coin mutters to itself "My friend is in trouble, I should try harder to come up tail this time to make it up to him." If you have ever overheard a coin saying this, then it makes sense to think that after 4 heads in a row, tails are overdue. But if you have not, it seems more rational to assume that a coin has no memory of the past.

By part (a), if we toss a coin 6 times, the probability of 3 or more heads and/or tails is about 0.59. The probability is already 1/2 if we toss a coin 5 times. Many people would find it surprising that the probability is so high. Intuitive estimates of probabilities are often wildly off. This gives the knowledgeable gambler many opportunities to make sucker bets.

In sports, if the local team loses three or four in a row, sports writers speculate darkly about collapse. Three or four wins in a row lead to dreams of winning the Cup. If a basketball player has a string of consecutive successful 3 point shots, the player is said to have a hot hand, and the string is expected to continue. Statistical analysis has shown that most sports streaks are probably not real, that underlying probabilities have not changed.

There are practical applications of these ideas. Your math teacher assigns as homework tossing a loonie 200 times and recording the results. You may not have a loonie, or may have alternate uses for your time. So you quickly jot down a "random" string of length 200 made up of the letters H and/or T.

Statistical analysis will with overwhelming likelihood expose the fraud. Your so-called random string will with high probability contain far too few medium to long strings of consecutive H or T, that is, far too many runs. If you have written down HHH, you will probably next write a T, because of a feeling that HHHH does not look random enough.

Runs-based analysis is widely used in detecting "anomalies" in reports of scientific experiments. The good news is that, by making use of a random number generator, you can cheat without risk of detection by such an analysis.

http://www.pims.math.ca/education/math_problems/
http://www.math.ubc.ca/~adler/problems/