Solutions to April 2008 Problems

Problem 1. Let $\mathcal{Q}$ be the convex quadrilateral whose vertices are the points where the curves with equations $x^4 + y^4 = 5$ and $xy = 1$ meet. Find as simple an *expression* as you can for the area enclosed by $\mathcal{Q}$.

Solution. Look at the system $x^4+y^4 = 5$, $xy = 1$. A rough drawing shows that there are 4 solutions. Note that if (a, b) is one solution, then the other solutions are (b, a), $(-a, -b)$, and $(-b, -a)$. Our quadrilateral is therefore a rectangle. We could compute the coordinates of the vertices explicitly, but it will turn out that there is no need to.

Using the Pythagorean Theorem, we find that the rectangle has sides of length $\sqrt{2(a-b)^2}$ and $\sqrt{2(a+b)^2}$. It follows that the rectangle has area $2\sqrt{(a-b)^2(a+b)^2}$. This is simply $2|a^2 - b^2|$.

Note that $a^4 + b^4 = 5$, and $a^2b^2 = 1$. It follows that

$$(a^2 - b^2)^2 = a^4 + b^4 - 2a^2b^2 = 3.$$

Thus $|a^2 - b^2| = \sqrt{3}$, and the rectangle has area $2\sqrt{3}$.

Problem 2. Let S be the set $\{1, 2, 3, 4, 5, 6, 7, 8, 9, 10\}$. How many ordered pairs (A, B) are there such that (i) A and B are subsets of S, and (ii) $A \cap B = \emptyset$ (the intersection of A and B is the empty set, that is, A and B have no element in common). Note that the empty set is a subset of any set.

Solution. There is a simple approach to the answer. Underneath each of 1, 2, 3, . . . , 10 write one of the letters a, b, or n. The letter a means that the number is being put into A, b means that it is being put into B, and n means that it is being put into neither. Every ordered pair (A, B) of the kind we are trying to count corresponds in this way to a uniquely determined 10-letter word over the alphabet $\{a, b, n\}$. So the number of ordered pairs is the same as the number of 10-letter words over a 3-letter alphabet. It follows that there are 3^{10} ordered pairs.

Another Way. Here is a more complicated approach. The set A could have 0, 1, 2, . . . , 10 elements.

We will count the number of ordered pairs (A, B) in which A has k elements. There are $\binom{10}{k}$ ways of choosing a k-element subset A of S. Any such A can be completed to a pair (A, B) of the desired kind by letting B be any subset of A'. (Here by A' we mean the set of elements of S which are not in A.)

Since A has k elements, it follows that A' has $10-k$ elements, so A' has 2^{10-k} subsets. It follows that there are $\binom{10}{k}2^{10-k}$ ways to make an ordered pair (A, B) of the right type such that A has k elements. We conclude that the full number of ordered pairs is

$$\sum_{k=0}^{10} \binom{10}{k} 2^{10-k}.$$

This is a fairly short sum, so with a little effort one could use it to compute a numerical answer. Note, however, that by the Binomial Theorem,

$$(2+1)^k = \sum_{1}^{10} \binom{10}{k} 2^{10-k},$$

so our expression simplifies to 3^{10}.

Another Way. Let S_n be the set $\{1, 2, 3, \dots, n\}$. Let $f(n)$ be the number of ordered pairs (A, B) such that A and B are subsets of S_n and A and B have no elements in common. We want to compute $f(10)$. But it is worthwhile to find a general expression for $f(n)$. We start with very small values of n.

Let $n = 1$. It is easy to list all the ordered pairs that work. They are $(\emptyset, \emptyset)$, $(\{1\}, \emptyset)$, and $(\emptyset, \{1\})$. It follows that $f(1) = 3$. We also take care of the case $n = 2$, by listing all possibilities for (A, B), and after a while conclude that $f(2) = 9$. On this admittedly thin evidence, we will conjecture that $f(n) = 3^n$ for every positive integer n.

Suppose that we know that $f(k) = 3^k$ for a particular k. We will show that $f(k+1) = 3^{k+1}$.

Look at an ordered pair (A, B) where A and B are subsets of S_k, and A and B have no element in common. Such a subset "gives birth" to an ordered pair (A', B') of subsets of S_{k+1} in 3 different ways. (i) We can let $A' = A$, and $B' = B$; (ii) We can let $A' = A \cup \{k+1\}$, and $B' = B$; (iii) We can let $A' = A$ and $B' = B \cup \{k+1\}$. It is fairly easy to see that in this way we obtain all ordered pairs (A', B') of subsets of S_{k+1} such that $A' \cap B' = \emptyset$. It follows that the number of such ordered pairs is 3×3^k, that is, 3^{k+1}.

We have shown that if $f(k) = 3^k$ then $f(k+1) = 3^{k+1}$. Thus $f(2) = 3^2$, and therefore $f(3) = 3^3$, and therefore $f(4) = 3^4$, and so on. It follows that $f(10) = 3^{10}$, and in general $f(n) = 3^n$.

Problem 3. Let $P(x)$ be a non-constant polynomial with real coefficients. Show that there is an irrational number θ such that $P(\theta)$ is irrational.

Solution. We will show something quite a bit stronger. Let K be the collection of all numbers of the form $m + n\sqrt{2}$, where m and n are integers, and $n \neq 0$. Since $\sqrt{2}$ is irrational, it is easy to show that every number in K is irrational.

We will show that for any non-constant polynomial $P(x)$, there is a number k in K such that $P(k)$ is irrational. Temporarily, a non-constant polynomial $P(x)$ will be called *bad* if $P(k)$ is rational for every k in K. We want to show that there are no bad polynomials.

First note that there is no bad polynomial of degree 1. For let $ax + b$, where $a \neq 0$, be a polynomial of degree 1. If $P(x)$ is bad, then $P(\sqrt{2})$ is a rational number r. Thus $a\sqrt{2} + b = r$. But since $P(x)$ is bad, $P(\sqrt{2} + 1)$ is a rational number s. Thus $a\sqrt{2} + a + b = s$. We conclude that $a = s - r$, so a is rational.

Since $P(x)$ is bad, we also have that $P(2\sqrt{2})$ is a rational number t. Thus $a(2\sqrt{2}) + b = t$. It follows that $a\sqrt{2} = t - r$. This is impossible, since $t - r$ is rational and a is a non-zero rational.

Next we deal with polynomials of degree greater than 1. We will show that if there is a bad polynomial of degree $d > 1$, then there is a bad polynomial of degree $d - 1$. Let $P(x)$ be a bad polnomial of degree greater than 1. Write $P(x)$ in the form

$$P(x) = a_0x^d + a_1x^{d-1} + \cdots + a_d.$$

Let $Q(x)$ be the polynomial $P(x + 1) - P(x)$. A short calculation shows that $Q(x)$ has degree $d - 1$. Furthermore, if m and n are integers, with $n \neq 0$, then

$$Q(m + n\sqrt{2}) = P(m + 1 + n\sqrt{2}) - P(m + n\sqrt{2}).$$

Since P is bad, each of $P(m+1+n\sqrt{2})$ and $P(m+n\sqrt{2})$ is rational, so $Q(m + n\sqrt{2})$ is rational, meaning that Q is bad.

So we have shown that if $P(x)$ is a bad polynomial of degree $d > 1$, we can construct another bad polynomial of degree $d - 1$. Do this "degree lowering" trick until we hit a bad polynomial of degree 1. But we showed that there is no bad polynomial of degree 1, so there cannot be a bad polynomial of any degree greater than 1.

Comment 1. We could instead give a short set-theoretic argument. Its shortness is, however, deceptive since the argument uses standard set-theoretic results that take a while to prove.

There are uncountably many reals, and countably many rationals. So the set of irrationals is uncountable. Let $P(x)$ be a polynomial of positive degree. For any fixed rational r, look at the set of all x such that $P(x) = r$. This set is finite, for the polynomial $Q(x) = P(x) - r$ has only finitely many roots. But the union of a countable collection of finite sets is countable. It follows that the set of numbers x such that $P(x)$ is rational is countable. In particular, this set cannot be all of the irrationals. So there are irrationals x such that $P(x)$ is irrational, in fact for "most" irrationals x, $P(x)$ is irrational.

Problem 4. Show that the sum of any number of consecutive perfect cubes is never prime.

Solution. The meaning of "perfect cube" is not completely clear. We could mean a positive integer which is the cube of an integer. Or else we could mean a non-negative integer which is the cube of an integer. Or else we could mean any integer, positive, negative, or 0, which is the cube of an integer. Actually, there is also some ambiguity about the meaning of the word "prime." The standard school meaning of the word is that a prime is an integer greater than 1 which has no positive integer divisors other than itself and 1. But we could let a prime be an integer p, such that $|p| > 1$, such that p has no integer divisors other than ± 1 and $\pm p$.

All these ambiguities have no real effect on our problem, since any sum of consecutive perfect cubes, under any interpretation, is either 0 or a sum of consecutive perfect cubes which are all of the same sign. So we will interpret "consecutive perfect cubes" to mean consecutive non-negative perfect cubes. And since 0 is not a prime, we can confine attention to sums of consecutive positive perfect cubes.

Our first argument uses the following result, which is interesting in itself.

Lemma. Let n be a positive integer. Then

$$1^3 + 2^3 + \cdots + n^3 = \left(\frac{n(n+1)}{2}\right)^2.$$

Proof. The usual argument is by induction. We give an argument which looks a little different from the standard one. Let $F(k) = ((k)(k+1)/2)^2$. A little algebra shows that for all positive k, $F(k) - F(k-1) = k^3$. It follows that

$$1^3 + 2^3 + \cdots + n^3 = (F(1) - F(0)) + (F(2) - F(1)) + \cdots + (F(n) - F(n-1)).$$

Look at the right-hand side above, and add up, noting the cancellations. We find that $1^3 + 2^3 + \cdots + n^3 = F(n) - F(0)$. Since $F(0) = 0$, we conclude that $1^3 + 2^3 + \cdots + n^3 = F(n)$, which is what we wanted to prove. □

Now look at a sum of positive consecutive cubes, say $(m+1)^3 + (m+2)^3 + \cdots + n^3$. This sum is equal to

$$(1^3 + 2^3 + \cdots + n^3) - (1^3 + 2^3 + \cdots + m^3).$$

Let $N = n(n+1)/2$, and let $M = m(m+1)/2$. By our lemma, the above difference is equal to $N^2 - M^2$, which factors as $(N+M)(N-M)$. It is easy to see that if $n > 1$, then neither of $N + M$ nor $N - M$ is equal to 1, so their product cannot be prime. And if $n = 1$, we must have $m = 0$, and our sum is 1, which is not a prime.

Another Way. Suppose that we are looking at the sum of an odd number of consecutive cubes. Let a^3 be the middle one, and let $2k+1$ be the number of cubes. Then our sum is equal to

$$(a-k)^3 + (a-k+1)^3 + \cdots (a-1)^3 + a^3 + (a+1)^3 + \cdots + (a+k-1)^3 + (a+k)^3.$$

For $j \geq 1$, look at the terms $(a-j)^3$ and $(a+j)^3$ symmterical about a^3. From the factorization $x^3 + y^3 = (x+y)(x^2 - xy + y^2)$, we conclude that $(a-j)^3 + (a+j)^3$ is divisible by $(a-j) + (a+j)$, and is therefore divisible by a. Thus our full sum is divisible by a, and is clearly greater than a unless $a = 1$. So our full sum is not prime.

More or less the same argument works for the sum of an even number of consecutive cubes. If the smallest cube is s^3 and the largest is t^3, the sum of the smallest and largest cubes is divisible by $s+t$. But the sum of the second smallest and second largest cubes, namely $(s+1)^3+(t-1)^3$, is divisible by $s+t$. Similarly, the sum of the third smallest term and the third largest term is divisible by $s+t$, and so on. So our full sum is divisible by $s+t$. This sum is clearly not equal to $s+t$, and $s+t>1$, so our full sum is not prime.

Problem 5. Draw a line with negative slope through the point $(4,2)$. Let X be the x-intercept of this line, let Y be the y-intercept, and let O be the origin. What is the smallest possible area of $\triangle OXY$? No calculus please.

Solution. Our first approach is algebraic. Take a line through $(4,2)$ with negative slope close to 0, and rotate the line clockwise about $(4,2)$ until it is nearly vertical. Initially, the triangle formed by the origin and the intercepts has large area, and at the end it again has large area. Somewhere in between, but perhaps at more than one place, the area reaches a minimum.

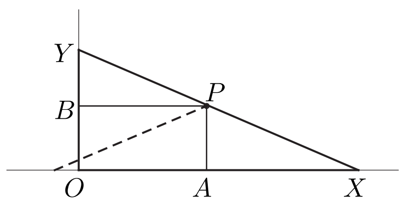
Suppose that our line has equation $y=-px+q$, where p is positive. The line goes through $(4,2)$, and therefore $q=2+4p$. Thus the line meets the y-axis at the point with y-coordinate $2+4p$. And a short calculation shows that the line meets the x-axis at the point with x-coordinate $(2+4p)/p$. It follows that $\triangle OXY$ has area $2(1+2p)^2/p$. We need to choose p so that the area is as small as possible. We will use the identity $(x+y)^2=(x-y)^2+4xy$. We find that

$$2\frac{(1+2p)^2}{p}=2\frac{(1-2p)^2}{p}+\frac{16p}{p}=2\frac{(1-2p)^2}{p}+16.$$

Since the term $2(1-2p)^2/p$ is non-negative, the area of our triangle is always ≥ 16. The area is equal to 16 for the line through $(4,2)$ that has slope equal to $-1/2$.

There are many variants of the same basic idea. Suppose for example that $OX=x$ and $OY=y$. A similar triangles argument, or a slope argument, shows that $y/x=2/(x-4)$. But the area of our triangle is $(1/2)xy$, that is, $x^2/(x-4)$. So we want to minimize $f(x)$, where $f(x)=x^2/(x-4)$, and $x>4$. It is convenient to let $t=x-4$. We then have $f(x)=(t+4)^2/t$. Now use the fact that $(t+4)^2=(t-4)^2+16t$ to conclude that $(t+4)^2/t=(t-4)^2/t+16$. It follows that our area is always greater than or equal to 16, and is 16 when $t=4$.

Another Way. This minimization problem can be solved geometrically. Let $P=(4,2)$. Draw a line through P, with intercepts X and Y, and let A and B be as in the diagram below.



Reflect $\triangle PXA$ in the line PA, and reflect $\triangle PBY$ in the line PB. The two reflected triangles cover rectangle $OAPB$, and possibly more. So the area of $\triangle OXY$ is at least twice the area of rectangle $OAPB$, with equality when $OA = AX$. But the rectangle has area 8, so the minimum area of the triangle is 16. The same argument works if P is any point with positive coordinates.