Solutions to April 2009 Problems

Problem 1. The National Basketball Association (NBA) has 30 teams; each plays 82 games during the regular season. Call the teams 1, 2, …, n, where $n = 30$. Let W_k be the number of regular season wins by team k, and let L_k be the number of regular season losses by team k. Show that

$$W_1^2 + W_2^2 + \cdots + W_n^2 = L_1^2 + L_2^2 + \cdots + L_n^2.$$

Solution. We will show that the difference between the left hand side and the right hand side is equal to 0. By rearranging terms slightly, we find that this difference is equal to

$$(W_1^2 - L_1^2) + (W_2^2 - L_2^2) + \cdots + (W_n^2 - L_n^2). \tag{1}$$

Each term $(W_i^2 - L_i^2)$ in the above sum can be factored as $(W_i + L_i)(W + i - L_i)$. But for any team i, the sum $W_i + L_i$ is the total number of games played by team i. Thus $W_i + L_i = 82$. It follows that the sum in Expression **??** is equal to

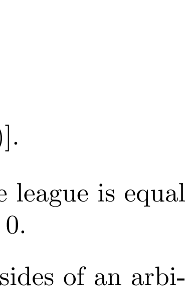
$$82[(W_1 - L_1) + (W_2 - L_2) + \cdots + (W_n - L_n)].$$

This sum is equal to

$$82[(W_1 + W_2 + \cdots + W_n) - (L_1 + L_2 + \cdots + L_n)].$$

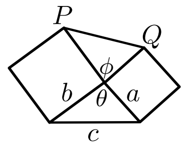
For every win there is a loss, so the total number of wins in the league is equal to the total number of losses. Hence the above sum is equal to 0.

Problem 2. Outward facing squares are erected on the three sides of an arbitrary triangle. Then pairs of vertices of the squares are joined as in the figure below, forming a convex hexagon. Show that if the sides of the original triangle are a, b, and c, then the sum of the squares of the sides of the hexagon is equal to $4(a^2 + b^2 + c^2)$.



Solution. Three of the 6 sides of the hexagon have, respectively, length a, b, and c. It remains to calculate the squares of the other 3 sides of the hexagon.

We look at a typical one of these 3 sides, say the side PQ in the figure below. Let θ be the angle between sides "a" and "b" in the original triangle, and let ϕ be as shown. Note that $\phi = 180° - \theta$.



By the Cosine Law,

$$c^2 = a^2 + b^2 - 2ab\cos(\theta).$$

Again by the Cosine Law, using the fact that $\cos(\phi) = -\cos(\theta)$, we have

$$(PQ)^2 = a^2 + b^2 + 2ab\cos(\theta).$$

Add. We obtain

$$(PQ)^2 = 2(a^2 + b^2) - c^2.$$

By symmetry, we can see that the squares of the remaining two sides of the hexagon are $2(b^2 + c^2) - a^2$ and $2(c^2 + a^2) - b^2$. Finally, add up. The sum of the 6 squares of the sides is $4(a^2 + b^2 + c^2)$.

Problem 3. For which positive integers n is $4^n + n^4$ prime?

Solution. When $n = 1$, $4^n + n^4$ is prime. We show that there is no other positive integer n such that $4^n + n^4$ is prime. If n is an even positive integer, then $4^n + n^4$ is obviously not prime. So let n be odd. It is easy to verify that

$$n^4 + 4^n = (n^2 - (2^{(n+1)/2})n + 2^n)(n^2 + (2^{(n+1)/2})n + 2^n), \qquad (2)$$

Since n is odd, $(n+1)/2$ is a positive integer, and therefore both $n^2-(2^{(n+1)/2})n+2^n$ and $n^2 + (2^{(n+1)/2})n + 2^n$ are integers.

It remains to show that if $n > 1$, then both of these factors of $n^4 + 4^n$ are greater than 1. That is obvious for $n^2 + (2^{(n+1)/2})n + 2^n$, so it remains to show that if n is an odd positive integer greater than 1, then $n^2 - (2^{(n+1)/2})n + 2^n$ is greater than 1. There are various ways to do this. We do it by a variant of completing the square. Note that

$$n^2 - (2^{(n+1)/2})n + 2^n = (n - 2^{n/2})^2 + (2^{n/2+1}n - n2^{(n+1)/2})n.$$

The first term on the right hand side of the above equation is clearly non-negative. The second term is $2^{n/2}(2 - \sqrt{2})n$, and this is clearly greater than 1 if $n > 1$.

Comment. It may seem as if Identity **??** was pulled out of a hat. Not quite: it is closely related to the interesting fact that the polynomial $x^4 + 4y^4$ factors nicely. This can be shown by "completing the square." We have

$$x^4 + 4y^4 = (x^2 + 2y^2)^2 - 4x^2y^2.$$

The expression on right hand side of the above equation is a difference of squares, so it factors as

$$(x^2 + 2y^2 - 2xy)(x^2 + 2y^2 + 2xy).$$

Problem 4. (a) Let $Q(x) = x^2 - x + 1$. Show that if m is an integer greater than 1 such that m divides a, then m divides none of $Q(a)$, $Q(Q(a))$, $Q(Q(Q(a)))$, and so on. (b) Use part (a) to prove that there are infinitely many primes.

Solution. (a) In order to avoid long strings of Q, let $a_0 = a$, $a_1 = Q(a_0)$, $a_2 = Q(a_1$, and so on. Suppose that m divides a_0. It follows that m divides $a_0^2 - a_0$, and therefore since $m > 1$, it cannot divide $a_0^2 - a_0 + 1$, that is, m cannot divide a_1.

In order to push to a_2 and beyond, we need to alter the strategy slightly. Note that we have proved above that if m divides a_0, then the remainder when a_1 is divided by m is equal to 1. Thus m divides $a_1 - 1$, and hence m divides $a_1(a_1 - 1)$. It follows that the remainder when $a_1^2 - a_1 +$ is divided by m is equal to 1. In particular, m does not divide a_2.

Since a_2 leaves a remainder of 1 on division by m, it follows in the same way that $a_2^2 - a_2 + 1$ (that is, a_3) leaves a remainder of 1 on division by m. And because a_3 leaves a remainder of 1 on division by m, so does a_4, and therefore so does a_5, and so on. Thus in particular if m divides a_0, then m cannot divide any a_i with $i > 0$.

(b) The argument of part (a) shows that if m divides a_i, then m cannot divide a_j for any $j \neq i$.

Let a_0 be any integer greater than 1. For example, let $a_0 = 2$. Consider the numbers a_0, a_1, a_2, and so on, where $a_{i+1} = Q(a_i)$. It is clear that all the a_i are greater than 1.

For any i, let p_i be a prime that divides a_i, for definiteness the smallest such prime. Since p_i divides a_i, then p_i cannot divide any a_j with $j \neq i$. In particular, if $j \neq i$, then $p_j \neq p_i$. It follows that the primes p_0, p_1, p_2, and so on are all different, so we have produced an infinite collection of primes.