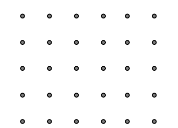
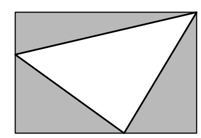
Solutions to April 2010 Problems

Problem 1. In the rectangular grid below, each point is at distance 1 from its horizontal and vertical neighbours. We pick three non-collinear grid points A, B, and C. What are all the possible values of the area of $\triangle ABC$?



Solution. Let $\mathcal{T}$ be a *lattice triangle,* that is, a triangle whose vertices have integer coordinates. Then there is a smallest rectangle with sides that are either vertical or horizontal, and that contains $\mathcal{T}$. We get a picture that (with some inessential variations) looks like this:



The rectangle has integer sides, and so has integer area. The three shaded triangles are right-angled, and have integer legs, so each has area of the shape $k/2$ where k is an integer. So the area of $\mathcal{T}$ (unshaded), which is the difference between the area of the rectangle and the area of the shaded region, is also of the shape $k/2$, where k is an integer.

Comment. There are many other ways to show that the area of a lattice triangle is of the form $k/2$, where k is an integer. Here is a very useful general result which we do not prove. Suppose that we are given three points, with coordinates (x_1, y_1), (x_2, y_2), and (x_3, y_3). Then the area of the triangle whose vertices are these three points is equal to

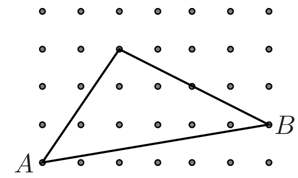
$$\frac{|x_1y_2 - y_1x_2 + x_2y_3 - y_2x_3 + x_3y_1 - y_3x_1|}{2}$$

It is clear from this formula that if the x_i and y_i are integers, then the area is of the form $k/2$ where k is an integer.

We should also mention here an attractive result usually called *Pick's Theorem.* Take any (not necessarily convex) *lattice polygon,* in the plane, that is, a polygon whose vertices have integer coordinates. Let i be the number of lattice points *inside* the polygon, and let b be the number of lattice points that are on the *boundary* of the polygon. Then the area of the lattice polygon is $i + b/2 - 1$.

We return to our specific problem about the 7×5 grid. It is easy to see that we cannot obtain any triangle with area greater than $(6)(4)/2$. And we already know that all achievable areas must have shape $k/2$ where k is an integer. We will now show that these are the only restrictions, that for all integers k where $1 \leq k \leq (6)(4)$, the area $k/2$ is achievable.

Let A and B be as in the diagram below, and let the third vertex ("C") roam freely over the lattice points that are "above" the line AB.



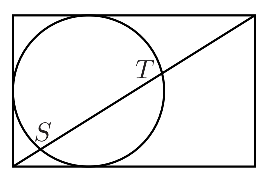
We will show that by choosing C appropriately, we can obtain any area $k/2$ where k is an integer and $1 \leq k \leq (6)(4)$.

Let the third vertex C travel along the lattice points on a horizontal line, for example the horizontal line where the third vertex is in the picture. If C is at the left end of the line, the triangle can be thought of as a triangle with base 3 and height 6, so area $(6)(3)/2$. If C is at the right end of the line, the triangle can be thought of as a triangle with base 2 and height 6, so area $(6)(2)/2$. As C travels from left to right, the height of the triangle, if we think of it as having base AB, steadily decreases. So area steadily decreases from $(6)(3)/2$ to $(6)(2)/2$. But at each lattice point the area is of the shape $k/2$ where k is an integer. There are 7 lattice points on the line, so 7 different areas. But there are only 7 numbers of the form $k/2$ between $(6)(3)/2$ and $(6)(2)/2$ (inclusive), so these must be all the areas of our triangles as C travels on the line.

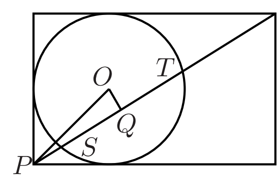
Similarly, if we let C travel on the top line, from left to right, we get as areas all the $k/2$ from $(6)(4)/2$ down to $(6)(3)/2$. As we travel along the third horizontal line from the bottom, we get in the same way all numbers of the form $k/2$ from $(6)(2)/2$ down to $(6)(1)/2$. And as we travel along the second horizontal line from the bottom, we get all numbers from $(6)(1)/2$ to $(6)(0)/2$. (The last number is not an area, unless we are willing to allow the "degenerate" triangle with $C = B$.) So the last non-zero number we get is $1/2$. So we can achieve as areas all $k/2$ from $(6)(4)/2$ down to $1/2$.

Comment. Essentially the same argument shows that if we have an $m \times n$ dot grid, then the triangles whose vertices are grid points can take on as areas all numbers $k/2$, where k ranges over the integers from $(m-1)(n-1)$ down to 1.

Problem 2. The picture is of a rectangle whose base is greater than its height. A circle is drawn in the rectangle, which is tangent to the top, left, and bottom edges. The diagonal from the bottom left corner to the top right corner meets the circle in points S and T. Express the distance ST in terms of the base and height.



Solution. Drop a perpendicular from the centre O of the circle to our diagonal line. Suppose that the perpendicular meets the diagonal at Q. Since Q bisects the segment ST, it is enough to compute SQ.



Let θ be the angle that the diagonal makes with the base, and let $\phi = \angle OPQ$. Then $\phi = \pi/4 - \theta$. But by the usual formula for the sine of a difference, we have

$$\sin\phi = (1/\sqrt{2})\cos\theta - (1/\sqrt{2})\sin\theta.$$

Since $\cos\theta = b/\sqrt{a^2+b^2}$ and $\sin\theta = a/\sqrt{a^2+b^2}$, we conclude that

$$\sin\phi = \frac{b-a}{\sqrt{2}\sqrt{a^2+b^2}}.$$

But $OQ = OP\sin\phi$, and OP is $\sqrt{2}$ times the radius a of the circle, so

$$OQ = \frac{a(b-a)}{\sqrt{a^2+b^2}}.$$

By the Pythagorean Theorem for $\triangle SOQ$, we have $(SQ)^2 = (SO)^2 - (OQ)^2$. Thus

$$(SQ)^2 = a^2 - \frac{a^2(b-a)^2}{a^2+b^2}.$$

This quickly simplifies to $2a^3b/(a^2+b^2)$. Take the square root, and double to find ST. It is $2a\sqrt{2ab}/\sqrt{a^2+b^2}$.

Another Way. We coordinatize, using the bottom left corner as origin, and the obvious two lines as axes.

Let the base be $2b$ and the height be $2a$. The circle has equation $(x-a)^2+(y-a)^2 = a^2$, and the line has equation $y = ax/b$. We begin to solve for the intersection points. To avoid fractions, it is convenient to multiply the equation through by a^2. We obtain

$$(ax-a^2)^2 + (ay-a^2)^2 = a^4.$$

Substitute by for ax. After some simplification we find that the y coordinates of S and T satisfy the equation

$$(a^2+b^2)y^2 - 2(a+b)(a^2)y + a^4 = 0.$$

In a similar way, we find that the x-coordinates satisfy the equation

$$(a^2+b^2)x^2 - 2(a+b)(ab)x + a^2b^2 = 0.$$

Let the coordinates of S be (u_1, v_1), and let the coordinates of T be (u_2, v_2). The square of the distance from S to T is given by $(u_1-u_2)^2 + (v_1-v_2)^2$. We could calculate u_1, u_2, and v_1, v_2, But it is neater to observe that

$$(u_1-u_2)^2 + (v_1-v_2)^2 = (u_1+u_2)^2 + (v_1+v_2)^2 - 4(u_1u_2 + v_1v_2).$$

The sum of the roots of our equation in x is $2(a+b)(ab)/(a^2+b^2)$, and the product of the roots is $a^2b^2/(a^2+b^2)$. Similarly, the sum of the roots of our equation in y is $2(a+b)(a^2)/(a^2+b^2)$ and the product is $a^4/(a+2+b^2)$. Calculate. We find that

$$(u_1+u_2)^2 + (v_1+v_2)^2 = 4a^2(a+b)^2/(a^2+b^2)$$

and

$$4(u_1u_2 + v_1v_2) = 4a^2 = 4a^2(a^2+b^2)/(a^2+b^2),$$

Now do the subtraction. We get that the square of the distance from S to T is $8a^3b/(a^2+b^2)$.

Problem 3. Define $F(n)$ by

$$F(n) = \left(1 + \frac{2}{3}\right)\left(1 + \frac{4}{9}\right)\left(1 + \frac{16}{81}\right)\cdots\left(1 + \frac{2^{2^n}}{3^{2^n}}\right).$$

Find the smallest number a such that $F(n) \le a$ for all n.

Solution. A calculator shows that for example at $n = 6$, the number $F(n)$ is very close to 3. So it is natural to guess that $a = 3$. But we need to *prove* this.

In the problem, replace $2/3$ with the number x. So

$$F(n) = (1 + x)\left(1 + x^2\right)\left(1 + x^4\right)\cdots\left(1 + x^{2^n}\right).$$

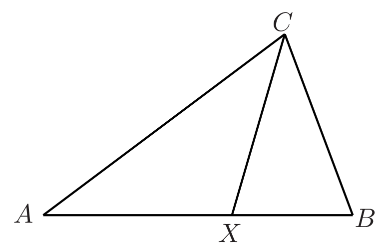
Multiply the left-hand side and the right-hand side by $1 - x$. Note that $(1 - x)(1 + x) = 1 - x^2$, and therefore $(1 - x)(1 + x)(1 + x^2) = 1 - x^4$, and so on. We conclude that

$$(1 - a)F(n) = 1 - x^{2^{n+1}}, \quad \text{so} \quad F(n) = \frac{1 - x^{2^{n+1}}}{1 - x}$$

(if $x \ne 1$). If $|x| > 1$, then $F(n)$ becomes very large positive as n gets large. But if $|x| < 1$, then $x^{2^{n+1}}$ approaches 0 (very rapidly if $x = 2/3$). Thus $F(n)$ approaches $1/(1-x)$. In the case $x = 2/3$, $F(n)$ is always below 3, but imperceptibly so if n is large. For example, already at $n = 6$ the number $F(n)$ is less than 10^{-22} below 3.

Problem 4. It is not difficult to verify that in the triangle with sides 4, 5, and 6, one of the angles is exactly twice another one. Find integers a, b, and c, which are sides of a triangle in which one angle is exactly twice another, such that the integers a, b, and c have no common divisor greater than 1, and all the integers are greater than 10.

Solution. In the figure below, ABC is a triangle, with $\angle C = 2\angle A$.



It is natural to bisect $\angle C$. Let the bisector meet AB at X. Let a, b, and c have their usual meanings, and let $XC = x$. Note that $XA = x$ and $XB = c - x$.

Note also that $\triangle CXB$ is similar to $\triangle ACB$. We conclude that

$$\frac{x}{b} = \frac{c - x}{a} = \frac{a}{c} \tag{1}$$

The first two equations yield $x = bc/(a + b)$, and then the first and third give $ab + b^2 = c^2$. (There are various more geometric ways to obtain the value of x, for example by extending AC by an amount a. Or else we can take advantage of the very useful geometric fact that if ABC is *any* triangle, then CX is the bisector of $\angle C$ if and only if $AX/AC = BX/BC$.)

Conversely, it is easy to verify that if $ab + b^2 = c^2$ then by putting $x = bc/(a + b)$ we can satisfy Equations 1, which implies that $\angle C = 2\angle A$.

Now we will use the equation $a(a + b) = c^2$ to identify all integer-sided triangles in which one angle is twice another. It is enough to look for triangles in which the greatest common divisor of a, b, and c is 1, for then the others can be found by multiplying each side by an integer scaling factor.

If $a(a + b) = c^2$, and a, b, and c have no common factor greater than 1, then in particular a and b cannot have a common factor greater than 1, that is, they must be relatively prime. For if d divides a and b, then d divides $a + b$, so d^2 divides c^2, which implies that d divides c. And of course if a and b have no common factor greater than 1, certainly a, b, and c cannot have a common factor greater than 1. So we are interested in (positive integer) solutions of the equation $a(a + b) = c^2$, where a and b are relatively prime. It is easy to see that a and b are relatively prime if and only if a and $a + b$ are relatively prime.

The product $a(a + b)$ is c^2, a perfect square,and a and $a + b$ are relatively prime, so $a = s^2$ and $a + b = t^2$ for some relatively prime integers s and t. This gives $a = s^2$, $b = t^2 - s^2$ and $c = st$. That almost gives the answer, except that we must make sure that we get a genuine triangle, that is, that the sum of any two sides is greater than the third. It is not hard to check that this is true precisely if $s < t < 2s$.

Now we can produce arbitrarily many examples. Recall that we wanted a, b, and c to be greater than 10. So for example let $s = 5$ and $t = 6$. That gives $a = 25$, $b = 11$, $c = 30$.