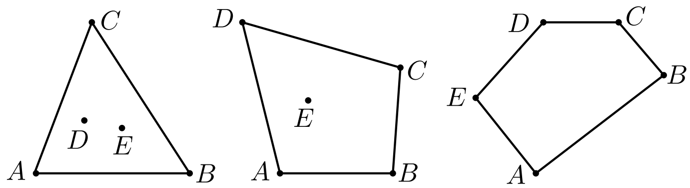
Solutions to April 2011 Problems

Problem 1. If A, B, C, D, and E are any 5 points in the plane, no 3 of which lie on a line, let $m(A, B, C, D, E)$ be the measure (in degrees) of the smallest angle determined by 3 of these points. What is the largest possible value of $m(A, B, C, D, E)$?

Solution. The 5 points fall into one of three basic patterns, which are illustrated below.



We introduce some terminology that comes up medium often in geometric problems. If S is a set of points in the plane or in space, we say that S is *convex* if given any points A and B in S, every point on the line segment AB is in S. If S is a set of points, then the *convex hull* of S is the "smallest" convex set that contains S. In our situation, let S be the set $\{A, B, C, D, E\}$. In the diagrams above, the convex hull of S is, from left to right, a triangle, a quadrilateral, and a pentagon.

Label points as in the pictures. Look first at the case where the convex hull of the 5 points is a triangle. At least one of the angles of the triangle, say the angle at C, is $\le 60°$. Join D and E to C. Imagine that the points are labelled so that D is "inside" $\angle ACE$. Then the sum of $\angle ACD$, $\angle DCE$, and $\angle ECB$ is equal to the angle at C. It follows that one of $\angle ACD$, $\angle DCE$, and $\angle ECB$ is $\le 20°$. Thus, for this sort of configuration, $m(A, B, C, D, E)$ is $\le 20°$.

Now suppose that the convex hull of the 5 points is a quadrilateral. At least one angle of the quadrilateral, say the angle at D, is $\le 90°$. By relabelling if necessary, we can assume that E is "inside" $\angle ADB$. Then $\angle ADE$, $\angle EDB$, and $\angle BDC$ have sum equal to the angle at D, which implies that one of the angles is $\le 30°$. Thus, for this sort of configuration, $m(A, B, C, D, E)$ is $\le 30°$.

Finally, suppose that the convex hull of the 5 points is a pentagon. At least one of the angles of the pentagon, say the angle at A, is $\le 108°$. But then the sum of $\angle EAD$, $\angle DAC$, and $\angle CAB$ is $\le 108°$, so at least one of the angles is $\le 36°$. Thus the largest possible value of $m(A, B, C, D, E)$ for this sort of configuration is $\le 36°$.

We show that the largest possible value of $m(A, B, C, D, E)$ for this sort of configuration is actually $36°$. Straightforward angle chasing shows that the smallest angle determined by 3 vertices of a regular pentagon is $36°$. Since when the convex hull is a triangle or a quadrilateral, the smallest angle is $\le 20°$ and $\le 30°$ respectively, the largest possible value of $m(A, B, C, D, E)$ over all configurations is $36°$.

Comment. Essentially this problem was Problem 2 of the 2011 Asian Pacific Mathematical Olympiad. It was unusually easy for an APMO Problem 2. (and Problem 1 may have been the easiest problem ever in a serious Olympiad.)

For the first two configurations, there is no reason to think that the bounds obtained are sharp. This leaves many potentially interesting questions. For example, for configurations like the middle one, what is the largest possible value of $m(A, B, C, D, E)$?

Problem 2. For any real number u, let $\{u\}$ be the *fractional part* of u, that is, $\{u\} = u - \lfloor u \rfloor$, where $\lfloor u \rfloor$ is the greatest integer which is less than or equal to u. Find all real numbers x such that $\{(x-1)^3\} = \{(x+1)^3\}$.

Solution. Our condition holds if and only if $(x+1)^3 - (x-1)^3$ is an integer a, that is, if and only if

$$6x^2 + 2 = a.$$

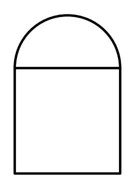
Let $b = a - 2$. So the real numbers x that work are given by $x = \pm\sqrt{b/6}$, where b ranges over the non-negative integers. Note that in particular this expression generates all integers, since $\pm n$ is obtained by taking $b = 6n^2$. Of course it was obvious anyway that all integers satisfied our relation. The point is that some non-integers also do. The simplest is $\sqrt{2}$, obtained from $b = 12$.

Problem 3. Every point in the plane is assigned one of two colours, red or blue. Show that if some positive real number does not occur as a distance between two red points, then *every* positive real number occurs as a distance between two blue points.

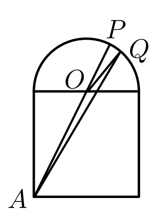
Solution. Suppose to the contrary that r is a missing red distance and b is a missing blue distance. Without loss of generality we may assume that $r \le b$. Since there is a missing red distance, not all points are red. Let B be a blue point. Let $C(B)$ be the circle (not the disk) with center B and radius b. Since the blue distance b is missing, all the points on $C(B)$ must be red. So for any distance d with $0 < d \le 2b$, there are two (red) points on $C(B)$ that are distance d apart. In particular there are two red points distance r apart.

Comment. If every point in the plane is assigned one of *three* colours, it turns out that there is a colour such that every real number occurs as a distance between two points of that colour. The proof is complicated, but not hopelessly so. It is known that there is a 6-colouring of the plane such that for any colour, there is a distance d which is not realized as the distance between two points of that colour. But possibly there is already a 4-colouring with this property. Despite considerable research effort, the problem remains open.

Problem 4. The *diameter* of a well-behaved region in the plane is the largest possible distance between two points in the region. What is the diameter of the region enclosed by the figure made up of a 1×1 square surmounted by a semicircle of diameter 1? Preferably, calculus should not be used.



Solution. It is clear that the largest possible distance is between the lower left corner of the square and somewhere on the right half of the semicircle (or by symmetry the lower right corner of the square and somewhere on the left half of the semicircle).

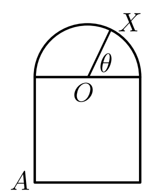


Let A be the lower left corner of the square, and let O be the center of the semicircle. Let the line through A and O meet the semicircle at P. Let Q be any other point on the semicircle. We will show that $AP > AQ$.

Note that $AP = AO + OP$. But $AO + OP = AO + OQ$, since OP and OQ are both radii. Now look at $\triangle AOQ$. The sum of any two sides of a triangle is greater than the third side (this is called the *Triangle Inequality.*) Thus $AP = AO + OQ > AQ$, which is what we wanted to prove.

The rest is an easy calculation. By the Pythagorean Theorem, $AO = \sqrt{1 + 1/4} = \sqrt{5}/2$. since $OP = 1/2$, it follows that the diameter is $(1 + \sqrt{5})/2$. This is a famous number, sometimes called the *golden number*, usually denoted by ϕ or τ. The golden number, or a relative, has the peculiar habit of showing up unexpectedly in the solution of mathematical problems.

Another Way. We explore approaches more typical of the methods used in calculus max/min problems, except that we will not use the calculus. The two solutions below are more painful than the first solution, but they introduce some useful ideas.



We try to avoid fractions by doubling the dimensions, computing, and dividing by 2 at the end. Let O be the center of the semicircle. Introduce coordinates, letting O be the origin, with the axes in their usual positions. Let A, X, and θ be as in the diagram. Then A has coordinates $(-1, -2)$ and X has coordinates $(\cos\theta, \sin\theta)$. The square of the distance from A to X is

$$(\cos\theta + 1)^2 + (\sin\theta + 2)^2.$$

After a little manipulation, this simplifies to $6 + 2(\cos\theta + 2\sin\theta)$.

We want to maximize the above expression, or equivalently to maximize $\cos\theta + 2\sin\theta$. We use a little trick to avoid the calculus. Rewrite $\cos\theta + 2\sin\theta$ as

$$\sqrt{5}\left(\frac{1}{\sqrt{5}}\cos\theta + \frac{2}{\sqrt{5}}\sin\theta\right).$$

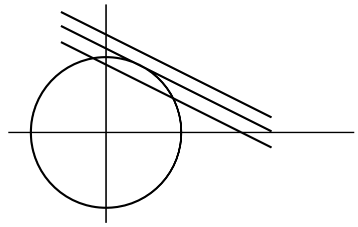
Let γ be the angle (between 0 and $\pi/2$) whose sine is $1/\sqrt{5}$ and whose cosine is therefore $2/\sqrt{5}$. Then our expression can be written as $\sqrt{5}\sin(\gamma + \theta)$.

This reaches a maximum of $\sqrt{5}$ when $\gamma + \theta = \pi/2$, that is, when θ is the angle (between 0 and $\pi/2$) whose sine is $2/\sqrt{5}$. So our squared distance has maximum value $6 + 2\sqrt{5}$. Scale back by multiplying distances by $1/2$. In the original problem, the maximum squared distance is $(3 + \sqrt{5})/2$. This is $((1 + \sqrt{5})/2)^2$. Thus our figure has diameter $(1 + \sqrt{5})/2$.

Another Way. Introduce coordinates like before, but show (by not doubling lengths) that we are not afraid of fractions Let X have coordinates (x, y). The square of the distance from A to X is

$$(x + 1/2)^2 + (y + 1)^2.$$

Every point on the semicircle satisfies the equation $x^2 + y^2 = 1/4$. We maximize the square of the distance subject to the condition $x^2 + y^2 = 1/4$. The square of the distance is $5/4 + x^2 + y^2 + x + 2y$, that is, $3/2 + x + 2y$. So we want to maximize $x + 2y$ subject to the condition $x^2 + y^2 = 1/4$.



Draw the graph of $x^2 + y^2 = 1/4$, and of the line $x + 2y = k$ for various values of k. As k increases, the line $x + 2y = k$ moves upward parallel to itself. The largest value of k for which the line $x + 2y = k$ meets $x^2 + y^2 = 1/4$ is the positive k for which the line is tangent to the circle. The tangent line has slope $-1/2$, so the line from the center of the circle to the point (x, y) of tangency is 2. Thus $y = 2x$. Together with $x^2 + y^2 = 1/4$ this implies that $x = 1/(2\sqrt{5})$ and $y = 2/(2\sqrt{5})$, so $x + 2y = \sqrt{5}/2$. Thus the maximum square of distance is $3/2 + \sqrt{5}/2$, and therefore the maximum distance is $(1 + \sqrt{5})/2$.

http://www.pims.math.ca/education/math_problems/
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