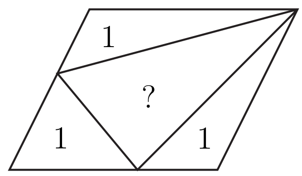
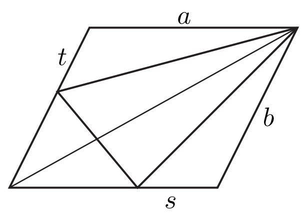
Solutions to May 2006 Problems

Problem 1. In the figure below, a parallelogram has been divided into four triangles. If the areas of the three “outer” triangles are each 1 as shown, what is the area of the fourth triangle?



Solution. It may look as if the data are not enough to determine the unknown area. But we will set up some equations and play around to see what happens. Let the dimensions of the triangles and parallelogram be as shown in the diagram below.



Let the area of the parallelogram be $2K$. The diagonal shown divides the parallelogram into two triangles of area K.

Look first at the triangle on the lower right. By comparing with the area of the half-parallelogram, we can see that it has area $(s/a)K$. Similarly, the triangle on the upper left has area $(t/b)K$. Both of these triangles have area 1. It follows that $a = sK$ and $b = tK$.

The triangle on the lower left has area $((a-s)/a)((b-t)/b)K$. From the fact that $a = sK$ and $b = tK$, the area simplifies to $((K-1)^2/K^2)K$. But this area is equal to 1. We therefore get $(K-1)^2 = K$, or equivalently $K^2 - 3K + 1 = 0$.

The quadratic equation has roots $(3 \pm \sqrt{5})/2$. One root is obviously too small, so $K = (3+\sqrt{5})/2$, and therefore the area of the parallelogram is $3+\sqrt{5}$. To find the area of the inner triangle, subtract the sum of the areas of the three outer triangles. We conclude that the inner triangle has area $\sqrt{5}$.

Comment. Suppose that the triangle at the lower left has area p, and the other two corner triangles have area q and r. The same argument as the one given above shows that the parallelogram has area

$$p + q + r + \sqrt{(p+q+r)^2 - 4qr}$$

and therefore the inner triangle has area $\sqrt{(p+q+r)^2 - 4qr}$.

In the particular case $p = q = r = 1$, we could simplify the calculations by multiplying distances in the y direction by a scaling factor k, and distances in the x direction by a scaling factor $1/k$, so that the lengths we called a and b become equal. This combination of transformations does not change areas. We do not need actually to compute k: it is enough to observe that without loss of generality $a = b$. This forces (in the notation of the diagram) $s = t$, and calculations are somewhat simplified.

Problem 2. Find the last non-zero digit of 1000!.

Solution. The following lemma is not strictly speaking needed, but is useful to know.

Lemma. Let n be a positive integer, and let p be prime. Then the largest integer k such that p^k divides $n!$ is given by

$$k = \left\lfloor \frac{n}{p} \right\rfloor + \left\lfloor \frac{n}{p^2} \right\rfloor + \left\lfloor \frac{n}{p^3} \right\rfloor + \cdots .$$

Proof. Here as usual $\lfloor x \rfloor$ denotes the greatest integer which is less than or equal to x. The sum in the statement of the lemma is finite, since if i is large enough, n/p^i is positive but less than 1, so $\lfloor n/p^i \rfloor = 0$.

Imagine that every integer i with $1 \le i \le n$ has to pay a fine of $f(i)$ dollars, where $f(i)$ is the largest integer such that $p^{f(i)}$ divides i. Then the largest k such that p^k divides $n!$ is the sum of the fines paid by all numbers i, as i ranges over the integers from 1 to n.

Collect the fine as follows. First collect 1 dollar from each multiple of p. There are $\lfloor n/p \rfloor$ multiples of p between 1 and n, so thus far we have collected $\lfloor n/p \rfloor$ dollars.

But some numbers may still owe money. For example, if $p^2 \le n$, it still owes 1 dollar. So collect 1 additional dollar from all multiples of p^2 that are in our interval. There are $\lfloor n/p^2 \rfloor$ such multiples, so at this second stage we collect $\lfloor n/p^2 \rfloor$ dollars.

The $\lfloor n/p^3 \rfloor$ multiples of p^3 still owe money. Collect 1 dollar from each. Continue in this way. The total fine collected is

$$\left\lfloor \frac{n}{p} \right\rfloor + \left\lfloor \frac{n}{p^2} \right\rfloor + \left\lfloor \frac{n}{p^3} \right\rfloor + \cdots ,$$

so this expression represents the largest k such that p^k divides $n!$. □

For example, let $n = 1000$ and $p = 5$. Then

$$k = \left\lfloor \frac{1000}{5} \right\rfloor + \left\lfloor \frac{1000}{25} \right\rfloor + \left\lfloor \frac{1000}{125} \right\rfloor + \left\lfloor \frac{1000}{625} \right\rfloor .$$

Calculate. We find that the largest power of 5 that divides 1000! is 5^{249}.

We can do a similar calculation with $n = 1000$ and $p = 2$. Note that the largest k such that 2^k divides 1000! is much larger than 249.

The above calculations tell us the number of 0's at the end of the decimal expansion of 1000!, that is, the largest k such that 10^k divides 1000!. Since there are fewer 5's "in" 1000! than there are 2's, this largest k is simply 249. Unfortunately, we were asked for the last non-zero digit in the decimal expansion of 1000!, not the number of terminal 0's. However, the 5's will still be crucial for the calculation.

Imagine multiplying together all the integers from 1 to 1000. These integers can be divided into five types. Type 0 consists of the integers not divisible by 5. Type 1 consists of the integers divisible by 5 but not by 25. Type 2 consists of the integers divisible by 25 but not by 125. Type 3 consists of the integers divisible by 125 but not by 625. And finally, 625 is the only integer of type 4.

We first examine the product of the integers of type 0. So we are looking at the product

$$(1 \cdot 2 \cdot 3 \cdot 4)(6 \cdot 7 \cdot 8 \cdot 9) \cdots (991 \cdot 992 \cdot 993 \cdot 994)(996 \cdot 997 \cdot 998 \cdot 999).$$

Each bracketed product has decimal expansion that ends in a 4, and there are 200 such products, so the full product is A_0, where the decimal expansion of A_0 ends in a 6.

Next look at the product of the integers of type 1. There are 160 such integers, namely 5, 10, 15, 20, 30, 35, 40, ..., 995. Their product is

$$5^{160}(1 \cdot 2 \cdot 3 \cdot 4)(6 \cdot 7 \cdot 8 \cdot 9) \cdots (191 \cdot 192 \cdot 193 \cdot 194)(196 \cdot 197 \cdot 198 \cdot 199).$$

This product is $5^{160}A_1$, where the decimal expansion of A_1 ends in a 6.

Similarly, we find that the product of the 32 integers of type 2 is

$$5^{64}(1 \cdot 2 \cdot 3 \cdot 4)(6 \cdot 7 \cdot 8 \cdot 9) \cdots (31 \cdot 32 \cdot 33 \cdot 34)(36 \cdot 37 \cdot 38 \cdot 39).$$

This product is $5^{64}A_2$, where the decimal expansion of A_2 ends in a 6.

Now look at the product of the 7 integers of type 3. This product is

$$5^{21}(1 \cdot 2 \cdot 3 \cdot 4)(6 \cdot 7 \cdot 8),$$

that is, $5^{21}A_3$, where the decimal expansion of A_3 ends in a 4. Finally, the "product" of the numbers of type 4 is 5^4. Thus

$$1000! = 5^{249}A_0A_1A_2A_3,$$

so $1000! = 5^{249}A$, where the decimal expansion of A ends in a 4.

Now we will grab 249 2's from A. They will join up with the 249 5's to produce the 249 terminal 0's in the decimal expansion of 1000!. Note that there are many more than 249 2's "in" A. Thus as we divide A 249 times by 2, we obtain an even number each time.

Recall that A "ends" with a 4. Thus $A/2$ must end with a 2 or a 7. But $A/2$ is even, so it ends with 2. Since $A/2$ ends with 2, $A/4$ must end with 1 or 6. But $A/4$ is even, so it ends with 6. Similarly, $A/8$ ends with 8, and $A/16$ ends with 4. Thus four cycles of division by 2 bring us back to the the same terminal digit 4. It follows that 248 cycles will bring us to terminal digit 4, and hence 249 cycles will bring us to terminal digit 2. We conclude that the last non-zero digit of 1000! is 2.

Comment. There is nothing special about the number 1000, though it is convenient that 1000 is divisible by 5^3, which means that its representation to the base 5 is quite simple. But the same basic strategy, with some computational complications, can handle similar problems for numbers whose expansion to the base 5 is more messy.

Problem 3. (a) Find numbers A, B, and C such that

$$\frac{1}{x(x+1)(x+2)} = \frac{A}{x} + \frac{B}{x+1} + \frac{C}{x+2}$$

for all $x \neq 0$, -1, or -2.

(b) Simplify:

$$\frac{1}{1 \cdot 2 \cdot 3} + \frac{1}{2 \cdot 3 \cdot 4} + \frac{1}{3 \cdot 4 \cdot 5} + \cdots + \frac{1}{99 \cdot 100 \cdot 101}.$$

Solution. (a) It is not clear that there *exist* numbers A, B, and C with the desired properties. We will use the following strategy. (i) *Assume* first of all that there *are* such numbers. (ii) On that assumption, find values of A, B, and C that work.

So suppose that for all x not equal to 0, -1, or -2, we have

$$\frac{1}{x(x+1)(x+2)} = \frac{A}{x} + \frac{B}{x+1} + \frac{C}{x+2}.$$

A little algebra (bringing to a common denominator) shows that we must then have

$$\frac{1}{x(x+1)(x+2)} = \frac{A(x+1)(x+2)}{x(x+1)(x+2)} + \frac{B(x)(x+2)}{x(x+1)(x+2)} + \frac{C(x)(x+1)}{x(x+1)(x+2)}. \quad (1)$$

for all x apart from the three forbidden values.

With a little manipulation, we can transform Equation 1 into

$$\frac{1}{x(x+1)(x+2)} = \frac{(A+B+C)x^2 + (3A+2B+C)x + 2A}{x(x+1)(x+2)} \quad (2)$$

(for all x except the forbidden values).

Now, *if* we can find A, B, and C such that

$$1 = (A+B+C)x^2 + (3A+2B+C)x + 2A \quad (3)$$

for all x, then certainly Equation 2 will hold for all x apart from the forbidden values, since the numerators on the left-hand side and right-hand side will match. (Indeed, Equation 1 will also in a sense hold at the forbidden values, for both sides will be undefined there.)

So Equation 1 will hold if

$$A+B+C = 0, \qquad 3A+2B+C = 0, \qquad \text{and} \qquad 2A = 1.$$

But it is easy to find A, B, and C that satisfy the system of three linear equations above. The third equation forces $A = -1/2$. Substitute in the first two equations. We obtain $B+C = -1/2$, $2B+C = -3/2$. This forces $B = -1$ and then $C = 1/2$.

Another Way. A variant of the above idea can save a considerable amount of computation in more complicated situations. Look again at Equation 2. This will hold if

$$1 = A(x+1)(x+2) + B(x)(x+2) + C(x)(x+1) \tag{4}$$

holds for all x. In the above equation, put $x = 0$. That yields $1 = 2A$, that is, $A = 1/2$. Put instead $x = -1$. That yields $B = -1$. Finally, put $x = -2$. That yields $C = 1/2$. Logically speaking, we are not finished, for that only shows that Equation 4 holds at these three particular values of x. But now it is easy to check by calculating that in fact Equation 4 holds for all x. Indeed we do not need to calculate. For the left-hand side of Equation 4 is a polynomial of degree less than or equal to 2, as is the right-hand side. And if two polynomials in x, each of degree less than or equal to n, agree at $n+1$ values of x, then they agree for all x.

(b) The terms that are being added are of the shape

$$\frac{1}{n(n+1)(n+2)}$$

where $n = 1$, 2, 3, and so on up to $n = 99$. By the result of part (a) the n-th term can be rewritten as

$$\frac{1}{2n} - \frac{1}{n+1} + \frac{1}{2(n+2)}.$$

Let S be the desired sum. It is marginally more convenient to find $2S$. We have

$$2S = \left(\frac{1}{1} - \frac{2}{2} + \frac{1}{3}\right) + \left(\frac{1}{2} - \frac{2}{3} + \frac{1}{4}\right) + \left(\frac{1}{3} - \frac{2}{4} + \frac{1}{5}\right) + \cdots + \left(\frac{1}{99} - \frac{2}{100} + \frac{1}{101}\right).$$

There is wholesale cancellation, if we bring things that have the same denominator together as follows:

$$\left(\frac{1}{1}\right) + \left(-\frac{2}{2} + \frac{1}{2}\right) + \left(\frac{1}{3} - \frac{2}{3} + \frac{1}{3}\right) + \left(\frac{1}{4} - \frac{2}{4} + \frac{1}{4}\right)$$
$$+ \left(\frac{1}{5} - \frac{2}{5} + \frac{1}{5}\right) + \cdots + \left(\frac{1}{99} - \frac{2}{99} + \frac{1}{99}\right) + \left(\frac{1}{100} - \frac{2}{100}\right) + \left(\frac{1}{101}\right).$$

We conclude that

$$2S = \frac{1}{2} - \frac{1}{100} + \frac{1}{101}, \quad \text{and} \quad S = \frac{1}{4} - \frac{1}{(2)(100)(101)}.$$

Further "simplification" is not worthwhile.

Another Way.

We can proceed more simply without making direct use of part (a). Note that in general

$$\frac{1}{n(n+1)(n+2)} = \frac{1/2}{n(n+1)} - \frac{1/2}{(n+1)(n+2)}.$$

Sum up from $n = 1$ to $n = 99$. Note the wholesale cancellation. We get $1/4 - (1/2)(1/100)(1/101)$.

Problem 4. Find all pairs (x, y) of real numbers that satisfy the two equations

$$x + y = 1$$
$$x^5 + y^5 = 11.$$

Solution. We could, using a graphing calculator or graphing program, plot the two curves of the problem, and read off an answer. Or else we could use some version of a "Solve" program to find a good numerical approximation to the solution. But since we can find an explicit closed form expression for the answer, such approximate procedures are really not adequate.

We explore for a while a "standard" approach. From the first equation, we find that $y = 1 - x$. Substitute $1 - x$ for y in the second equation. We get $x^5 + (1 - x)^5 = 11$. Expand $(1 - x)^5$, either by using the Binomial Theorem or by patiently multiplying out. After some simplification we arrive at $x^4 - 2x^3 + 2x^2 - x - 2 = 0$. This equation has no obvious solution, maybe.

The equation is of degree 4 in x. There *is* a method for solving fourth-degree equations, one that goes back to the sixteenth-century mathematicians Cardano and Ferrari. But the method is complicated, so we abandon this approach.

It is better to exploit the symmetry in the problem, either by keeping the variables x and y, or by making a symmetry-preserving substitution. So if we are going to substitute, it seems wiser to note that if $x + y = 1$, then $x = 1/2 + t$ and $y = 1/2 - t$ for some real number t. Substitute in the second equation. We get

$$(1/2 + t)^5 + (1/2 - t)^5 = 11.$$

Expand. There is substantial cancellation, and after a while we arrive at

$$16t^4 + 8t^2 - 35 = 0.$$

This is a quadratic equation in t^2. To put it another way, if we let $w = t^2$ we get a quadratic equation in w.

The quadratic equation can be solved for t^2 by using the Quadratic Formula. We get

$$t^2 = \frac{-8 \pm \sqrt{64 + 2240}}{32}.$$

But t is real, so t^2 can not be negative, and after some calculation we arrive at

$$t^2 = \frac{5}{4},$$

and conclude that $t = \pm\dfrac{\sqrt{5}}{2}$. Alternately, we could observe that

$$16t^4 + 8t^2 - 35 = (4t^2 - 5)(4t^2 + 7)$$

and therefore $t^2 = 5/4$.

Finally, we can write down the real solutions of the original system of equations. They are

$$(x, y) = \left(\frac{1+\sqrt{5}}{2}, \frac{1-\sqrt{5}}{2}\right) \quad \text{and} \quad (x, y) = \left(\frac{1-\sqrt{5}}{2}, \frac{1+\sqrt{5}}{2}\right).$$

Comment. Of course we cheated in setting up the problem. The "11" in the equation $x^5 + y^5 = 11$ was chosen to make numbers turn out nicely, but not *too* nicely. However, even if we use a number other than 11, the argument does not change much. If for example we are given the equations $x + y = a$, $x^5 + y^5 = b$, we let $x = a/2 + t$; thus $y = a/2 - t$. When we substitute into $x^5 + y^5 = b$ and simplify, we obtain an equation which is quadratic in t^2. The numerical details change, but the structure does not.

Early in the proof, we remarked that $x^4 - 2x^3 + 2x^2 - x - 2 = 0$ has no obvious solution. That is not quite true. We *might* notice that the equation can be rewritten as $(x^2 - x - 1)(x^2 - x + 2) = 0$, and then everything is easy. If, however, we start with $x^5 + y^5 = b$, factorization is not at all obvious.

Another Way. We can keep both variables around for a while. There are many ways to do this. For example, we could use the fact that

$$x^5 + y^5 = (x + y)(x^4 - x^3y + x^2y^2 - xy^3 + y^4)$$

and therefore $x^4 - x^3y + x^2y^2 - xy^3 + y^4 = 11$.

But $x^2 + y^2 = (x + y)^2 - 2xy$. Let $p = xy$. Then $x^2 + y^2 = 1 - 2p$. Note now that $x^4 + y^4 = (x^2 + y^2)^2 - 2x^2y^2 = (1 - 2p)^2 - 2p^2$. Also, $x^3y + xy^3 = xy(x^2 + y^2) = p(1 - 2p)$. We therefore arrive at the equation

$$[(1 - 2p)^2 - 2p^2] + p^2 - [p(1 - 2p)] = 11.$$

After a little manipulation we arrive at $5p^2 - 5p - 10 = 0$. This quadratic equation has the roots $p = 2$ and $p = -1$. So we have the two possibilities $x + y = 1$, $xy = 2$, and $x + y = 1$, $xy = -1$.

Note that $(x - y)^2 = (x + y)^2 - 4xy = 1 - 4p$. This shows that $p = 2$ is impossible, since it would give $(x - y)^2 = -7$. If $p = -1$, we get $(x - y)^2 = 1 - 4p = 5$. That gives $x - y = \pm\sqrt{5}$. Finally, using $x + y = 1$ and adding and subtracting, we find x and y.

There are many variants of this argument. For example, note that

$$x^5 + y^5 = (x^2 + y^2)(x^3 + y^3) - x^2y^2(x + y).$$

But

$$x^2 + y^2 = (x + y)^2 - 2xy \qquad \text{and} \qquad x^3 + y^3 = (x + y)^3 - 3xy(x + y).$$

Now let $p = xy$. Using the fact that $x + y = 1$ we find

$$11 = x^5 + y^5 = (1 - 2p)(1 - 3p) - p^2.$$

The above equation simplifies to $5p^2 - 5p - 10 = 0$, and again we find that $p = 2$ or $p = -1$.

Comment. Each method exploited symmetry. Often the only feasible way to solve a problem is to take advantage of symmetries. That is frequently true even in very applied problems. For example, the fact that gravitation is centrally symmetric is a key fact in deducing the motions of the planets. If there is symmetry, it is usually important to hold on to it as long as possible.

Problem 5. Call a set $\mathcal{S}$ of positive integers *multiple-rich* if for any positive integer n, some multiple of n (perhaps n itself) is in $\mathcal{S}$. For example, the set of positive perfect squares is multiple-rich, and the set of primes is not.

Suppose that the positive integers are divided into two teams, say $\mathcal{A}$ and $\mathcal{B}$. Show that at least one of $\mathcal{A}$ or $\mathcal{B}$ is multiple-rich.

Solution. Suppose that $\mathcal{A}$ is *not* multiple-rich. Then there is a positive integer r such that $\mathcal{A}$ does not contain any multiple of r.

Similarly, if $\mathcal{B}$ is not multiple-rich, there is a positive integer s such that $\mathcal{B}$ does not contain any multiple of s.

Let $t = rs$. Note that t is simultaneously a multiple of r and a multiple of s. So t cannot be in either $\mathcal{A}$ or $\mathcal{B}$. This is impossible, since $\mathcal{A}$ and $\mathcal{B}$ between them contain *all* the positive integers.

Thus the assumption that *neither* $\mathcal{A}$ nor $\mathcal{B}$ is multiple-rich leads us to something that cannot be true. It follows that this assumption is false, meaning that *at least* one of $\mathcal{A}$ or $\mathcal{B}$ is multiple-rich.

Comment. Note that more or less the same argument shows that if $\mathcal{S}$ is multiple-rich, and is divided into finitely many teams, then at least one of the teams is multiple-rich.

Was this problem easy or hard? The proof was short, much shorter than the other arguments in this problem set. But it can *feel* harder: there are no formulas to manipulate.

The argument is indirect, what is sometimes called "proof by contradiction." We proved that the result holds by showing that assuming it does not hold leads to a conclusion that is demonstrably false, an absurdity, a contradiction.

"Proof by contradiction" is a basic tool in mathematics. The standard proofs of many famous results use this strategy. Two important examples are the proof that there are infinitely many primes, and the proof that $\sqrt{2}$ is not a rational number.

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