Solutions to May 2007 Problems

Problem 1. A circle meets the parabola $y = x^2$ at four points. The x-coordinates of three of the points are 2, 3, and 4. Find the x-coordinate of the fourth point.

Solution. A natural approach is to suppose that the circle has center (a, b) and radius r, and calculate these three numbers.

The points $(2, 4)$, $(3, 9)$, and $(4, 16)$ are equidistant from (a, b). It follows that

$$(2 - a)^2 + (4 - b)^2 = (3 - a)^2 + (9 - b)^2 = (4 - a)^2 + (16 - b)^2.$$

The first equation simplifies to $a+5b = 35$, and the second simplifies to $a+7b = 91$. Solve. It turns out that $b = 28$ and $a = -105$.

The circle has equation

$$(x + 105)^2 + (y - 28)^2 = r^2.$$

Put $x = 2$ and $y = 4$. We conclude that $r^2 = 12025$. After some simplification, we find that our circle has equation

$$x^2 + y^2 + 210x - 56y - 216 = 0.$$

To find where the circle meets the parabola, substitute x^2 for y. We obtain

$$x^4 - 55x^2 + 210x - 216 = 0.$$

Three of the roots of this equation are known, so it is easy to find the fourth. A complicated way of doing it is to divide the polynomial on the left by $(x - 2)(x - 3)(x - 4)$. It is much easier to note that the coefficient of x^3 is 0, and therefore the sum of the roots is 0. Since the sum of three of them is 9, the fourth must be -9.

Another Way. Almost all of the calculations were unnecessary! The circle has an equation of the shape $x^2 + y^2 + cx + dy + e = 0$. *Imagine* substituting x^2 for y in the above equation. We get an equation of the form $P(x) = 0$, where $P(x)$ is a polynomial of degree 4, and the coefficient of x^3 is 0. It follows that the sum of the roots of $P(x)$ is 0, so if we know three roots we can easily find the fourth. We don't need to know c, d and e.

Problem 2. In how many ways can one distribute 6 apples, 7 bananas, and 8 cantaloupes between 10 people? All fruit of the same type are identical. Fruit must remain whole, must all be given out, and possibly Alphonse gets everything.

Solution. We solve a somewhat more general problem. How many ways are there to distribute a apples, b bananas, and c cantaloupes between n people?

We were told that fruit of the same kind are indistinguishable. So as long as Alphonse gets 2 apples, it does not matter *which* two apples. But people are distinguishable, so 2 apples and 1 banana for Alphonse, and 7 cantaloupes for Beth, is different from 7 cantaloupes for Alphonse, and 2 apples and 1 banana for Beth. (Alphonse doesn't like bananas.)

We distribute the apples first. For concreteness in this part of the discussion, assume that there are 6 apples and 10 people. Let us do the distributing in the following strange way. Distribute 16 (not a typo!) apples between the 10 people, in such a way that everyone gets at least 1 apple. Then take away 1 apple from everybody. It is clear that there are exactly as many ways to distribute 6 apples among 10 people, with no restriction, as there are to distribute 16 apples among 10 people, with everyone receiving at least 1 apple.

Now we count the number of ways of distributing 16 apples among 10 people so that everyone gets at least 1 apple. Lay out the 16 apples in a row, like this:

$$\bigcirc \quad \bigcirc \quad \bigcirc \quad \bigcirc \quad \bigcirc \quad \bigcirc \quad \bigcirc \quad \bigcirc \quad \bigcirc \quad \bigcirc \quad \bigcirc \quad \bigcirc \quad \bigcirc \quad \bigcirc \quad \bigcirc \quad \bigcirc$$

Call the space between successive apples a *gap.* Note that there are 15 gaps. *Choose* 9 of these gaps, and put a "separator," maybe a raisin, in each of the chosen gaps. Arrange our 10 people alphabetically. The first person in the list gets all the apples from the left end of the row to the first raisin. The next person in the list gets all the apples from the first raisin to the second raisin, and so on.

We conclude that the number of ways of distributing 16 apples among 10 people, so that everyone gets at least one apple, is $\binom{15}{9}$, also variously known as ${}_{15}\mathrm{C}_9$ or $\mathrm{C}(15, 9)$ or C_9^{15}. And by our earlier discussion, this is also the number of ways of distributing 6 apples among 10 people, with no restrictions.

Exactly the same argument shows that the number of ways of distributing a apples among n people (where people may get different numbers of apples) is $\binom{n+a-1}{n-1}$.

Once we have distributed the apples, we distribute the bananas. For every way of distributing the apples, there are $\binom{n+b-1}{n-1}$ ways to distribute the bananas. Finally, for every way of distributing the apples and bananas, there are $\binom{n+c-1}{n-1}$ ways to distribute the cantaloupes.

We conclude that the number of ways of distributing a apples, b bananas, and c cantaloupes among n people is

$$\binom{n+a-1}{n-1}\binom{n+b-1}{n-1}\binom{n+c-1}{n-1}.$$

The idea generalizes readily to the situation where we must also distribute d doughnuts (man cannot live by fruit alone).

For the particular numbers in our problem, we find that the number of ways of distributing the fruit is

$$\binom{15}{9}\binom{16}{9}\binom{17}{9}.$$

I think this turns out to be 1391922532000.

Problem 3. For any positive integer n, let $f(n)$ be the largest odd divisor of n. Evaluate the sum

$$f(1001) + f(1002) + f(1003) + \cdots + f(1999) + f(2000).$$

Solution. We can patiently calculate $f(1001)$, $f(1002)$, and so on up to $f(2000)$, and add. The computations are easy. After a (long) while we find that our desired sum is 1000000. There must be a better way!

The numbers in our problem are large, and there are a lot of them. So we look at smaller problems. Let N be a positive integer, and let

$$g(N) = f(N+1) + f(N+2) + \cdots + f(2N).$$

Our problem asks for $g(1000)$. We will compute $g(N)$ for a few values of N much smaller than 1000.

For example, let $N = 1$. Note that $g(1) = f(2) = 1$. Let $N = 2$. Note that $g(2) = f(3)+f(4) = 3+1 = 4$. Let $N = 3$. Note that $g(3) = f(4)+f(5)+f(6) = 1+5+3 = 9$. Let $N = 4$. Note that $g(4) = f(5) + f(6) + f(7) + f(8) = 5+3+7+1 = 16$.

It is not unreasonable to conjecture that $g(N) = N^2$ for any positive integer N. This conjecture is strengthened if we compute $g(5)$, $g(6)$, and $g(7)$, which indeed turn out to be 25, 36, and 49. But this information about small numbers N says nothing certain about $g(1000)$.

Let's look more closely at our computation of $g(3)$. Note that $f(4)$, $f(5)$, and $f(6)$ are the odd numbers 1, 3, 5 (in a different order). Similarly, $f(5)$, $f(6)$, $f(7)$, and $f(8)$ are the numbers 1, 3, 5, and 7 (in a different order). We will show that the numbers $f(1001)$, $f(1002)$, $f(1003)$, $\ldots$, $f(2000)$ are the numbers 1, 3, 5, $\ldots$, 1999 in some order, and more generally that for any positive integer N, the numbers $f(N+1)$, $f(N+2)$, $\ldots$, $f(2N)$ are 1, 3, 5, $\ldots$, $2N-1$ in some order. The collection of numbers $f(N+1)$, $f(N+2)$, $f(N+3)$, $\ldots$, $f(2N)$ consists of odd numbers in the interval from 1 to $2N-1$. We will show that these numbers are *all* of the N odd numbers from 1 to $2N-1$.

Let a be any odd integer in the interval from 1 to $2N-1$. If $a \geq N+1$, do nothing. Otherwise, multiply a by 2. If the result is greater than or equal to $N+1$, do nothing. Otherwise, multiply the result by 2. Continue in this way. After a while we must reach an integer b in the interval $[N+1, 2N-1]$. It is obvious that $f(b) = a$.

Thus every odd number a in the interval $[1, 2N-1]$ is equal to $f(b)$ for some b in $[N+1, 2N]$. There are N odd numbers a in the interval $[1, 2N-1]$, and N numbers in the interval $[N+1, 2N]$, so the numbers $f(a)$, where $N+1 \leq a \leq 2N$, must coincide with the N odd numbers from 1 to $2N-1$.

Finally, we show that for any positive integer N, $g(N) = N^2$ by showing that

$$1 + 3 + 5 + \cdots + (2N-1) = N^2.$$

This is a standard fact. There are a number of proofs, of which we give only one.

Note that in general $x^2 - (x-1)^2 = 2x - 1$. In particular, $1^2 - 0^2 = 1$, $2^2 - 1^2 = 3$, $3^2 - 2^2 = 5$, $4^2 - 3^2 = 7$, and so on. It follows that

$$\begin{aligned}&1 + 3 + 5 + \cdots + (2N-1)\\ =&(1^2 - 0^2) + (2^2 - 1^2) + (3^2 - 2^2) + \cdots + (N^2 - (N-1)^2).\end{aligned}$$

Look at the second sum. Note the wholesale cancellations: the sum collapses to N^2.

Problem 4. (a) Points P and Q are chosen on the curve $x^2 + 4y^2 = 1$ in such a way that the distance PQ is as large as possible. Find that distance. (b) Solve the same problem for the curve $x^4 + 16y^4 = 1$.

Solution. (a) The curve $x^2 + 4y^2 = 1$ is an ellipse that meets the x-axis at $(\pm 1, 0)$ and the y-axis at $(0, \pm 1/2)$. It is obvious that the two points farthest apart are $(1, 0)$ and $(-1, 0)$, so the maximum distance is 2. Is it really obvious? Only if ellipses look like their usual pictures. In fact, book representations of ellipses are often wrong. So to be certain we work a little more.

The problem is about *two* points P and Q. First we get rid of one of them. Let P and Q be two points on the ellipse, and let O be the origin. Without loss of generality we may assume that $OP \geq OQ$. Let P' be the reflection of P across the origin. Then $PP' \geq PQ$. So we need only find a point P that is as far from the origin as possible and then let $Q = P'$.

Let $P = (x, y)$. We want to maximize $x^2 + y^2$ given that $x^2 + 4y^2 = 1$. But

$$x^2 + y^2 = (x^2 + 4y^2) - 3y^2 = 1 - 3y^2,$$

and $1 - 3y^2$ reaches a maximum when $y = 0$. Thus the largest achievable distance is indeed 2.

(b) The curve $x^4 + 16y^4 = 1$ looks somewhat like an ellipse, and meets the axes exactly where $x^2 + 4y^2 = 1$ does. But the maximum distance is *not* 2. We can argue exactly as in (a) that it is enough to find a point P at maximum distance from the origin. We now maximize $x^2 + y^2$ given that $x^4 + 16y^4 = 1$.

Let $x^2 = u$ and $4y^2 = v$. We want to maximize $u + v/4$ given that $u^2 + v^2 = 1$. This is just Problem 4 of March 2997, with different numbers, and any of the four solutions of that problem works. For example, we can argue that we need to find $k > 0$ such that the line $u + v/4 = k$ is tangent to the circle $u^2 + v^2 = 1$.

Let R be the point of tangency. The line $v = 4(k - u)$ has slope -4. The line joining the origin to R is perpendicular to the tangent line, so has slope $1/4$, and therefore equation $v = u/4$. This line meets $u^2 + v^2 = 1$ at $(4/\sqrt{17}, 1/\sqrt{17})$, so $k = \sqrt{17}/4$. Thus the largest distance between points on our curve is $\sqrt{17}/2$, a little more than 2.

Comment. Consider the curve $|x|^e + |2y|^e = 1$, where e is a positive constant. If $e = 2$ we get the ellipse of (a), and if $e = 4$ the curve of (b). It turns out that if $e \leq 2$, then the largest distance between points on the curve is 2, but if $e > 2$, then the largest distance is greater than 2. The exponent 2 is right on the boundary. So maybe the fact that 2 is the largest distance in the ellipse isn't so obvious after all!

Problem 5. Let α, β, and γ be the solutions of $x^3 - 3x + 1 = 0$. Find

$$(2\alpha - 1)(2\beta - 1)(2\gamma - 1).$$

Solution. By sketching $y = x^3 - 3x + 1$, or otherwise, we can see that the given equation has three real roots. If the problem came up in a practical setting, we could use a graphing program, or some other method, to produce good numerical estimates of the roots, and use these estimates to calculate an approximate answer. But we can also find an *exact* answer.

Expand the given expression. We obtain

$$8(\alpha\beta\gamma) - 4(\alpha\beta + \beta\gamma + \gamma\alpha) + 2(\alpha + \beta + \gamma) - 1.$$

Note that

$$x^3 - 3x + 1 = (x - \alpha)(x - \beta)(x - \gamma). \tag{1}$$

By expanding the right-hand side, and comparing terms, we find that $\alpha+\beta+\gamma = 0$, $\alpha\beta + \beta\gamma + \gamma\alpha = -3$, and $\alpha\beta\gamma = -1$. Substitute. We conclude that our expression is equal to 3.

Another way: Let $f(x) = x^3 - 3x + 1$. Then by Equation **??**, $(-2)^3 f(x)$ is identically equal to

$$(-2)^3 f(x) = (2\alpha - 2x)(2\beta - 2x)(2\gamma - 2x).$$

Let $x = 1/2$. We conclude that

$$(2\alpha - 1)(2\beta - 1)(2\gamma - 1) = (-2)^3 f(1/2) = 3.$$

Comment. There are ways (originally due to Cardano and, in another version, to Viète) to find exact expressions for the roots of the cubic, but that approach is much more complicated.

In general, expanding like we did in the first solution isn't a good idea, since it makes things look more complicated. But in this case the terms $\alpha + \beta + \gamma$, $\alpha\beta + \beta\gamma + \gamma\alpha$, and $\alpha\beta\gamma$ that turn up can be read off from the coefficients of $x^3 - 3x + 1$, so the penalty for expanding is minor.