Solutions to May 2008 Problems

Problem 1. Alphonse and Beti work together to separate suckers from their money. Beti leaves the room. The mark (Mark) is asked to toss a loonie and a toonie. Alphonse hands one of the tossed coins back to Mark, and bets Mark $20 that Beti can guess whether that coin landed head or tail. Beti comes back in, and guesses correctly. How? If the game is played many times, how can Alphonse and Beti make it harder for Mark to know *how* he is being fleeced?

Solution. We describe a simple strategy that works, which we call strategy A. If the two coins both show heads, or both show tails, then Alphonse gives the loonie to Mark. If loonie shows head, while the toonie shows tail, or vice-versa, then Alphonse gives the toonie to Mark. It is now easy for Beti, by glancing at the remaining coin, to see what the result on the other coin must have been.

Strategy B runs along the same lines, except that if the two coins show the same thing, then Alphonse gives the toonie to Mark, and if the results on the two coins are different, Alphonse gives the loonie to Mark.

Strategies A and B can be combined to make it hard for Mark to know how he is being fleeced. Before the fleecing begins, Alphonse and Beti get together and toss repeatedly a fair coin with one side labelled A and the other side labelled B. They each memorize the sequence thus obtained. Suppose for example that the sequence begins with AABABB. Then in the first game with Mark, Alphonse and Beti use strategy A, in the second game they use strategy A, in the third game they use strategy B, in the fourth they use strategy A, and so on. This ensures that there is no apparent connection between the coin given to Mark and Beti's call.

Problem 2. For any real number u, let $\lfloor u \rfloor$ be the greatest integer which is less than or equal to u. For example, $\lfloor 17/5 \rfloor = 3$. How many integers n in the interval from 0 to 999 (inclusive) can be expressed in the form

$$n = \lfloor x \rfloor + \lfloor 2x \rfloor + \lfloor 3x \rfloor + \lfloor 4x \rfloor,$$

where x is a real number?

Solution. Let x be a non-negative real number. It is easy to see that

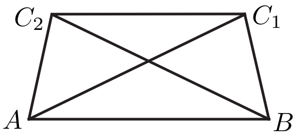
$$\lfloor (x+1) \rfloor + \lfloor 2(x+1) \rfloor + \lfloor 3(x+1) \rfloor + \lfloor 4(x+1) \rfloor = \lfloor x \rfloor + \lfloor 2x \rfloor + \lfloor 3x \rfloor + \lfloor 4x \rfloor + 10.$$

It follows that the non-negative integer n has a representation of the type we are looking for if and only if $n + 10$ has such a representation. Thus, to find out how many integers in the interval from 0 to 999 are representable, all we need to do is to find out how many integers from 0 to 9 are representable, and multiply the result by 100.

Let $f(x) = \lfloor x \rfloor + \lfloor 2x \rfloor + \lfloor 3x \rfloor + \lfloor 4x \rfloor$. To find out which integers from 0 to 9 are representable, we study $f(x)$ as x ranges over the interval $[0, 1)$. Imagine that x increases from 0 towards 1. Then $f(x)$ changes value at $x = 1/4$, $x = 1/3$, $x = 1/2$, $x = 2/3$, and $x = 3/4$, so $f(x)$ takes on precisely 6 values as x ranges over the interval $[0, 1)$ (these values happen to be 0, 1, 2, 4, 5, and 6). It follows that precisely 600 of the integers in the interval $[0, 999]$ can be expressed in the desired form.

Problem 3. A triangle of area 1 has sides a, b, and c, where $a \le b \le c$. Find the least possible value of b.

Solution. For a while, imagine that c is fixed. We are then looking at all triangles ABC, with $AB = c$, such that AB is a largest side of $\triangle ABC$, and such that the height of $\triangle ABC$, with AB viewed as the base, is equal to $2/c$. In the figure below, we have $AB = AC_1 = BC_2 = c$.



It is clear that if C is to be chosen "above" AB so that $\triangle ABC$ has area 1, and AB is a largest side of $\triangle ABC$, then C must lie on the line segment C_1C_2. It is also reasonably clear from the picture (and could be proved, if we were being fussy), that the second smallest side of such a triangle ABC is smallest if C lies exactly halfway between C_1 and C_2.

So we can assume that our target triangle ABC has $a = b \le c$. Now look at all such triangles with fixed value of a (and hence b). It is easy to see that such a triangle has largest area if $\angle BCA$ is a right angle, for this gives maximum height over base AC. Such a triangle has area $a^2/2$. So if the triangle is to have area 1, it follows that we must have $a = b = \sqrt{2}$.

Problem 4. Find all ordered pairs (a, b) of integers such that $a^2 - 4b$ and $b^2 - 4a$ are both perfect squares. In this kind of problem, one needs to *prove* that all solutions have been found.

Solution. First look at the case $a = 0$. Then $-4b$ must be a perfect square. This happens precisely if $b = -n^2$ for some integer n. That gives us the ordered pairs $(0, -n^2)$, where n ranges over all integers. By symmetry, we also have the solutions $(-n^2, 0)$, where n ranges over all integers.

Now we look for solutions (a, b) where neither a nor b is equal to 0. There are unfortunately a few cases to consider. Look first at the case where a and b are both negative.

So $a^2 - 4b$ is a perfect square bigger than a^2. Note that $a^2 - 4b$ has the same parity as a^2, that is, they are both even or both odd. It follows that $a^2 - 4b$ is at least equal to $(|a| + 2)^2$. We conclude that

$$a^2 - 4b \geq (|a| + 2)^2.$$

A little manipulation shows that $-4b \geq 4|a| + 4$, so $|b| \geq |a| + 1$. By symmetry, we also have $|a| \geq |b| + 1$. But it is not possible for both of these inequalities to hold.

Now look at the case where $a > 0$ and $b < 0$. Just as before, we conclude that $|b| \geq a + 1$. Now use the fact that $b^2 - 4a$ is a perfect square. So since $b^2 - 4a$ is a perfect square with the same parity as b^2, we have

$$b^2 - 4a \leq (|b| - 2)^2.$$

A little manipulation now shows that $|b| \leq a + 1. Itfollowsthat$—b—=a+1. Thus we must have $b = -(a + 1)$ where a is positive. Suppose conversely that $b = -(a+1)$. Then $a^2 - 4b = a^2 + 4a + 4$, a perfect square, and $b^2 - 4a = a^2 - 2a + 1$, a perfect square. Thus this case yields the family of solutions $(n, -(n+1))$, where n ranges over the positive integers.

By symmetry, the case $a < 0$, $b > 0$ has the family of solutions $(-(n+1), n)$, where n ranges over the positive integers.

Finally, we examine the possibility $a > 0$, $b > 0$. By reasoning of the type we have used above, since $a^2 - 4b$ and $b^2 - 4a$ are perfect squares we must have

$$a^2 - 4b \leq (a - 2)^2 \qquad \text{and} \qquad b^2 - 4a \leq (b - 2)^2.$$

These inequalities simplify to $b \geq a - 1$ and $a \geq b - 1$. It follows that $a = b + 1$, or $a = b + 1$, or $a = b$. It is easy to verify that if a is any positive integer, and $b = a + 1$, then the pair (a, b) satisfies our conditions, and by symmetry also b is any positive integer and $a = b + 1$, then (a, b) satisfies our conditions. This gives the solutions $(n, n + 1)$, and $(n + 1, n)$, where n ranges over the positive integers.

It remains to worry about the case where a and b are positive and equal, which has not been ruled out by our inequalities. Let them both be equal to c. We want $c^2 - 4c$ to be a perfect square. First imagine that c is odd. Then c and $c - 4$ have no common divisor greater than 1. Since $c^2 - 4c = c(c - 4)$, it follows that c and $c - 4$ are both perfect squares. But there is no odd c with this property. Now imagine that c is even, say $c = 2d$. Then $c(c - 4) = 4d(d - 2)$. So $d(d - 2)$ is a perfect square. An argument like the one just used shows that d cannot be odd. So we must have $d = 2e$ for some positive integer e. It follows that $c(c - 4) = 16e(e - 1)$. Thus e and $e - 1$ must be both perfect squares. This happens only if $e = 1$. So we conclude that the only solution with a and b positive and equal is $(4, 4)$.

Comment 1. It is not hard to find all the solutions we have described, and to check that they work. What took some effort here is the argument that there cannot be any other solutions.

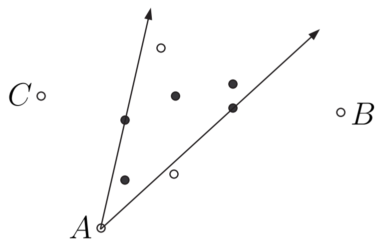
Problem 5. We are given a set $\mathcal{S}$ of 200 points in the plane, no three collinear. Of these points, 100 are white, and 100 are black. Show that there exists a collection of 100 line segments such that (i) the endpoints of any line segment lie in $\mathcal{S}$ and are of different colours and (ii) no two of the line segments meet.

Solution. We prove the result for any set of $2n$ points in the plane, no three collinear, with n of these points white and n black. The proof will be by induction, in this case a kind of "divide and conquer" induction. The result is trivial if $n = 1$. We will now show that if the result if true for all sets with $2k$ elements, with $k < n$, then the result is true for all sets with $2n$ elements.

Take our set $\mathcal{S}$ of $2n$ elements. Two points P and Q of $\mathcal{S}$ will be said to be the endpoints of a *boundary edge* of $\mathcal{S}$ if all points of $\mathcal{S}$ apart from P and Q lie on the same side of the line through P and Q.

Suppose first of all that there are points P and Q which are endpoints of a boundary edge of $\mathcal{S}$ and which are of different colours. Join these two points by a line segment PQ. There are then $2n - 2$ points of $\mathcal{S}$ left, half white and half black. Thus by the induction assumption these can be joined by $n - 1$ line segments that join points of opposite colour, and that do not meet. These line segments clearly do not meet PQ, so these line segments, together with PQ, do the desired job.

We now look at the case where all points which are endpoints of a boundary edge are of the same colour, say white. This situation is illustrated by the picture below.



In the picture, A, B, and C are consecutive endpoints of boundary edges of $\mathcal{S}$, with A between B and C. Imagine drawing the half-line that starts at A and passes through B. Now rotate this half-line counterclockwise. Every time the rotated line passes through a black point, stop to think and count. The picture shows the line as it passes through the first black point it meets during its rotation, and also shows the line as it passes through the last black point.

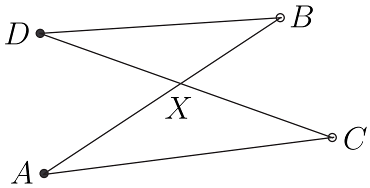
When the line meets the first black point in its rotation, it has left behind at least one white point, namely B (actually, in the picture as drawn it has left two white points behind). When the rotating line meets the last black point, there are more black points behind it than there are white points. Since black points are met one at a time, it follows that at some point the rotating line has exactly as many black points as white points behind it. Let us freeze it in that situation.

So our line goes through A and some black point X. On each side of the line through A and X, there is equality between the number of black points and the number of white points. Join A and X. By the induction assumption, the points on one side of the line can be joined up in white-black pairs so that the line segments so determined do not meet. And of course these line segments do not meet the segment AX. The same sort of pairing can be done on the other side of the line through A and X, so we have obtained the desired full pairing.

Another Way. The next solution is far less "natural," but in some ways much simpler. Imagine dividing our 200 points into 100 white-black pairs. If $\mathcal{P}$ is any such pairing, let the *length* of $\mathcal{P}$ be the sum of all distances AB, where A is white, B is black, and the set $\{A, B\}$ is a pair in $\mathcal{P}$. So the length of $\mathcal{P}$ is just the combined length of the line segments that we draw.

There are only finitely many ways to divide our 200 points into 100 white-black pairs (indeed there are 100! such ways, but that is not important here). Call such a division $\mathcal{Q}$ into pairs *minimal* if the length of $\mathcal{Q}$ is less than or equal to the length of $\mathcal{P}$ for every division $\mathcal{P}$ into pairs. Since there are only finitely many pairings, there is a minimal pairing (there might be several).

Now let $\mathcal{Q}$ be a minimal pairing. We show that no two line segments in $\mathcal{Q}$ can meet, that is, $\mathcal{Q}$ does the job. Suppose to the contrary that $\mathcal{Q}$ is a minimal pairing, and that two of its line segments meet, as in the picture below.



In the picture, A is black, B is white, and A and B are paired; C is white, D is black, and C and D are paired. And the line segments AB and CD meet at X. We will show this is impossible, by showing that this situation contradicts the assumption that $\mathcal{Q}$ is a minimal pairing.

For modify the pairing $\mathcal{Q}$ to a new pairing $\mathcal{Q}'$ by pairing A with C and B with D. We will show that this modified pairing is cheaper that $\mathcal{Q}$. The contribution of the points A, B, C, D to the length of $\mathcal{Q}$ is $AB + CD$, which is equal to

$$AX + XB + CX + XD \qquad \text{that is} \qquad (AX + CX) + (XB + XD).$$

However, by the Triangle Inequality, we have $AX + CX < AC$ (the sum of the two sides of a triangle is less than the third side), and again by the triangle inequality $XB + XD < BD$. It follows that the contribution of the points A, B, C, and D to the length of $\mathcal{Q}'$ is less than their contribution to the length of $\mathcal{Q}$. Thus the length of $\mathcal{Q}'$ is less than the length of $\mathcal{Q}$, contradicting the assumed minimality of the length of $\mathcal{Q}$. This completes the argument.

`http://www.pims.math.ca/education/math_problems/`
`http://www.math.ubc.ca/~adler/problems/`