Solutions to May 2009 Problems

Problem 1. Suppose that

$$P(x) = \frac{a}{3!}x(x-1)(x-2) + \frac{b}{2!}x(x-1) + \frac{c}{1!}x + d.$$

Show that $P(n)$ is an integer for every integer n if and only if all of a, b, c, and d are integers.

Solution. We show first that $x(x-1)/2!$ and $x(x-1)(x-2)/3!$ are both integers for every integer x. If x is an integer, then one of x or $x-1$ is even, and it follows immediately that $x(x-1)$ is divisible by 2, so $x(x-1)/2!$ is an integer. Also, one of x, $x-1$, or $x-2$ is divisible by 3, so $x(x-1)(x-2)$ is divisible by 2 and by 3, and therefore $x(x-1)(x-2)/3!$ is an integer.

Comment. For any real (or indeed complex) number x, and any non-negative integer k, define $\binom{x}{k}$ by

$$\binom{x}{k} = \frac{x(x-1)(x-2)\cdots(x-k+1)}{k!}.$$

Note that if x is a non-negative integer, and k is a non-negative integer that is $\leq x$, then $\binom{x}{k}$ as defined above is the ordinary binomial coefficient $\binom{x}{k}$.

We show that $\binom{x}{k}$ is an integer for any integer x. For if x is an integer which is $\geq x$, then $\binom{x}{k}$ is an integer, since it is the number of ways of choosing k objects from x objects. If x is non-negative and less than k, then $\binom{x}{k} = 0$, so $\binom{x}{k}$ is an integer. If $k = 0$, then we are dividing by $0!$, that is, by 1, so we obtain an integer. Finally, suppose that x is negative and k is positive. Let $y = |x|$. Then

$$\binom{x}{k} = \frac{(-y)(-y-1)\cdots(-y-k+1}{k!} = \pm\binom{y+k-1}{k},$$

and $\binom{y+k-1}{k}$ is an integer.

Since $\binom{x}{k}$ is an integer when x is an integer and $k = 0$, 1, 2, or 3, it follows that if a, b, c, and d are integers, then $P(n)$ is an integer for every integer n.

Next we prove the converse: If $P(n)$ is an integer for every integer n, then a, b, c, and d are integers.

From the fact that $P(0)$ is an integer, it follows immediately that d is an integer. From the fact that $P(1)$ is an integer, it follows that $c+d$ is an integer, and thus so is c. From the fact that $P(2)$ is an integer, it follows that $b+2c+d$

is an integer, and therefore b is an integer. Finally, from the fact that $P(3)$ is an integer, it follows that $a + 3b + 3c + d$ is an integer, and therefore a is an integer.

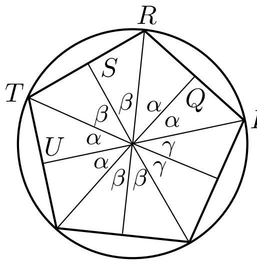
Comment. Essentially the same argument shows that if $P(x)$ is the polynomial given by

$$P(x) = a_0\binom{x}{n} + a_1\binom{x}{n-1} + a_2\binom{x}{n-2} + \cdots + a_n\binom{x}{0},$$

then $P(n)$ is an integer for every integer n if and only if the a_i are all integers.

Problem 2. Let n be odd. Show that if an n-gon inscribed in a circle has all its angles equal, then all its sides are equal.

Solution. We illustrate what happens with $n = 5$, but the argument is general. Consider the picture below.



Let O be the center of the circle that the n-gon is inscribed in. Join O to each of the vertices of the n-gon, and drop perpendiculars from O to each side of the n-gon.

Look for example at $\triangle OPQ$ and $\triangle ORQ$. These are congruent, and therefore $\angle POQ = \angle ROQ$ (these angles are both marked α in the diagram). Similarly, we have $\angle ROS = \angle TOS$, and so on (these are marked β in the diagram).

Now look at quadrilaterals $OQRS$ and $OSTU$. Since $\angle QRS = \angle STU$, it follows that $\angle QOS = \angle SOU$. We can then conclude that $\angle TOU = \alpha$. The same argument shows that as we go counterclockwise around the circle, angles are α, α, β, β, α, α, and so on. Thus the angles marked as γ are equal to α, since they follow counterclockwise from angles marked β. But they also follow clockwsie from angles marked α, and therefore $\gamma = \beta$. We can therefore conclude that $\alpha = \beta$, so our n-gon is regular, and therefore all its sides are equal.

Comment. The argument breaks down if n is even. And indeed it is easy for any even n to produce an inscribed n-gon with equal angles but unequal sides. The simplest example is a say 2×1 rectangle. But the argument can be adapted to show that if n is even, then alternate sides must be equal.

Problem 3. Call a triangle *accidental* if its area (in units^2) is numerically equal to its perimeter (in units). (a) What is the smallest possible value of the perimeter of an accidental right-angled triangle? (b) What is the smallest possible value of the perimeter of an accidental triangle?

Solution. (a) Imagine a right triangle whose two legs are x and y. Setting the perimeter equal to the area, we obtain the equation

$$x + y + \sqrt{x^2 + y^2} = \frac{xy}{2}.$$

Simplifying and rearranging, we obtain

$$2\sqrt{x^2 + y^2} = xy - 2(x + y).$$

Square both sides. We obtain

$$4x^2 + 4y^2 = (xy)^2 - 4xy(x + y) + 4x^2 + 8xy + 4y^2.$$

After some cancellation and division by xy, we obtain

$$xy - 4x - 4y = -8.$$

By a procedure analogous to "completing the square," this may be rewritten as

$$(x - 4)(y - 4) = 8.$$

We want to minimize the area, or equivalently to minimize xy. Since $xy = 4x+4y-8$, this is equivalent to minimizing $x+y$, or equivalently $(x-4)+(y-4)$. Let $u = x-4$ and $v = y-4$. We want to minimize $u+v$ subject to the condition $uv = 8$.

This is a standard problem, to which there are many approaches. We first note that we have the further condition that u and v are positive. This is somewhat hidden by the algebra done earlier. Recall that we had

$$2\sqrt{x^2 + y^2} = xy - 2(x + y)$$

and squared both sides. This hid the fact that the left-hand side, and therefore the right-hand side, must be positive. Thus $xy - 2(x + y)$ is positive, or equivalently $(x - 2)(y - 2) > 4$. Since x and y are positive, this forces $x > 2$ and $y > 2$. But then the fact that $(x - 4)(y - 4) = 8$ forces $x > 4$ and $y > 4$. So u and v are positive. Since

$$(u + v)^2 = (u - v)^2 + 4uv = (u - v)^2 + 32,$$

it follows that $u+v$ is minimal when $u-v = 0$. This means that $x = y = 4+\sqrt{8}$. The smallest possible area (and therefore perimeter) of an accidental right-angled triangle is therefore $12 + 8\sqrt{2}$.

Another Way. We first prove a lemma which is more general than necessary.

Lemma. Among all triangles XYZ such that $\angle XZY = \theta$, and such that the area of $\triangle XYZ$ is a given number A, the isosceles triangle with $ZY = ZX$ has the smallest perimeter.

Proof. Let $x = ZY$ and $y = ZX$. Then $A = (1/2)xy\sin\theta$, and $XY = \sqrt{x^2+y^2-2xy\cos\theta}$. So we are trying to minimize

$$x + y + \sqrt{x^2+y^2-2xy\cos\theta}.$$

Since A and θ are fixed, xy is fixed. By an argument given in the first solution, $x+y$ is minimized when $x = y$. Also, $2xy\cos\theta$ is fixed. So $\sqrt{x^2+y^2-2xy\cos\theta}$ is minimized when x^2+y^2 is. But x^2y^2 is fixed, so x^2+y^2 is minimized when $x^2 = y^2$. It follows that the perimeter is minimized when $x = y$. □

Now let I be the isosceles right-angled triangle whose perimeter is numerically equal to its area. If the legs of I have length x, the perimeter is $2x + \sqrt{2}x$, and the area is $x^2/2$. So $2(2+\sqrt{2})x = x^2$, and therefore $x = 2(2+\sqrt{2})$. A short calculation shows that the area (and perimeter) are equal to $12 + 8\sqrt{2}$.

We will show that among all right triangles with perimeter numerically equal to area, I has the smallest perimeter. Let T be a right triangle with perimeter numerically equal to area. Let the area of T be A. We will show that if T is not isosceles, then the area of T is greater than the area of I, that is, greater than $12 + 8\sqrt{2}$.

For let W be the isosceles right triangle with area A. Since W is isosceles and has the same area as T, the above lemma can be used to conclude that the perimeter p of W is less than the perimeter of T, so $p < A$.

Now scale W by the scale factor p/A. Then the perimeter of W scaled is $p(p/A)$ and the area of W scaled is $A(p/A)^2$, so they are numerically equal, and therefore W scaled is congruent to I. It follows that $p(p/A) = 12 + 8\sqrt{2}$. Since $p < A$, it follows that $A > 12 + 8\sqrt{2}$. This completes the proof.

Comment. The first argument was shorter, but the second one seems more natural. The same idea can be used to compute the minimum possible perimeter of an accidental triangle one of whose angles has a given value θ.

(b) The solution will use the fact that of all triangles with given area, the equilateral triangle has minimum perimeter, or equivalently that of all triangles with given perimeter, the equilateral triangle has maximum area. These are "well-known" facts. It is not hard to find informal justificaltion for them, but a formal argument takes more effort.

One proof uses the ever-useful Arithmetic Mean Geometric Mean (AM–GM) inequality, which we state but do not prove.

Lemma (AM–GM)**.** Suppose that a_1, a_2, $\ldots a_n$ are positive. Tnen

$$\frac{a_1 + a_2 + \cdots + a_n}{n} \geq (a_1 a_2 \cdots a_n)^{1/n},$$

with equality only when all the a_i are equal.

Now take a triangle with variable sides a, b, and c but fixed perimeter, which we denote by $2s$. By Heron's formula, the area of the triangle is

$$\sqrt{s(s-a)(s-b)(s-c)}.$$

To maximize the area, we need to maximize $(s-a)(s-b)(s-c)$. By the AM–GM inequality, we have

$$((s-a)(s-b)(s-c))^{1/3} \leq \frac{(s-a)+(s-b)+(s-c)}{3} = \frac{s}{3},$$

with equality only when $s-a = s-b = s-c$, that is, only when the triangle is equilateral. This shows that for given perimeter, the equilateral triangle has maximum area.

After having completed these preliminaries, we turn to the (much less interesting) problem of finding the minimum perimeter (or equivalently, area) of an accidental triangle.

Let E be the equilateral triangle whose perimeter is numerically equal to its area. If a is length of a side of E, then the perimeter of E is $3a$, and the area of E is $a^2\sqrt{3}/4$. If perimeter is numerically equal to area, then $a = 4\sqrt{3}$, and the perimeter is equal to $12\sqrt{3}$.

We will show that among all triangles with perimeter numerically equal to area, E has the smallest perimeter. Let T be a triangle with perimeter numerically equal to area. Let the perimeter of T be p. We will show that if T is not equilateral, then the area of T is greater than the area of E, and hence the perimeter of T is greater than $12\sqrt{3}$.

For let W be the equilateral triangle with perimeter p. Since W is equilateral and has the same perimeter as T, and T is not equilateral, we conclude that the area A of W is greater than the area of T, so $A > p$.

Now scale W by the scale factor p/A, which is less than 1. Then the perimeter of W scaled is $p(p/A)$ and the area of W scaled is $A(p/A)^2$, so they are numerically equal, and hence $p(p/A) = 12\sqrt{3}$. But $p/A < 1$, so $p > 12\sqrt{3}$. This completes the proof.

Comment. Any question that features a perimeter numerically equal to an area makes a mathematician shudder. This is because this "property" obviously depends on the unit of measurement being used. If the area (in square metres) is numerically equal to the perimeter (in metres), then the area (in cm^2) is 100 times the perimeter (in cm). Thus the property of being "accidental" is not at all geometric, it is truly just an accident caused by the choice of units. My excuse for posing the problem is that it came up in a Math teachers' discussion site, and that useful ideas can be found in the solution.

Problem 4. Let the sequence a_0, a_1, a_2, and so on be defined by $a_0 = 1$, and

$$a_0a_n + a_1a_{n-1} + a_2a_{n-2} + \cdots + a_na_0 = 1$$

for all $n > 0$. Find an explicit expression for a_n.

Solution. It is often useful to compute a little, in the hope that something interesting turns up. Let $n = 1$. Our equation becomes $a_0a_1 + a_1a_0 = 1$, and thus $a_1 = 1/2$. Let $n = 2$. Our equation becomes $a_0a_2 + a_1a_1 + a_2a_0 = 1$, which yields $a_2 = 3/8$. Let $n = 3$. Our equation becomes $a_0a_3 + a_1a_2 + a_2a_1 +$

$a_3a_0 = 1$, which yields $a_3 = 5/16$. Now put $a = 4$. Our equation becomes $a_0a_4 + a_1a_3 + a_2a_2 + a_3a_1 + a_4a_0 = 1$. Thus $a_4 = 35/128$. Finally, put $n = 5$. We find that $a_5 = 63/256$. Nothing really jumps out, except perhaps the powers of 2.

Maybe one could look at successive ratios. We have $a_1/a_0 = 1/2$, $a_2/a_1 = 3/4$, $a_3/a_2 = 5/6$, $a_4/a_3 = 7/8$, $a_5/a_4 = 9/10$. There is an apparent pattern. We have $a_2 = (1/2)(3/4)$, $a_3 = (1/2)(3/4)(5/6)$, $a_4 = (1/2)(3/4)(5/6)(7/8)$, $a_5 = (1/2)(3/4)(5/6)(7/8)(9/10)$, and (maybe) so on. It is reasonable, though perhaps premature, to conjecture that $a_n = (1 \cdot 3 \cdots (2n-1))/(2 \cdot 4 \cdots (2n))$. Multiply top and bottom of the conjectured expression by $(2 \cdot 4 \cdots (2n))$, and note that $(2 \cdot 4 \cdots (2n)) = (2^n)(n!)$. Now let

$$b_n = \frac{1}{2^{2n}}\binom{2n}{n}.$$

We want to show that $a_n = b_n$ for all n. The given recurrence equation clearly determines a_n once we know all the a_k for $k < n$. We have already verified that $a_k = b_k$ for $k = 0$, 1, 2, 3, 4, and 5. So we need only show that

$$b_0b_n + b_1b_{n-1} + b_2b_{n-2} + \cdots + b_nb_0 = 1.$$

I had an argument in mind for this that turned out to be mistaken. The solution given below will depend on properties of generating functions. In a sense the argument is quite natural, but unfortunately it depends on ideas that go beyond standard school mathematics.

We first state without proof the generalized Binomial Theorem, which seems to be due to Isaac Newton.

Theorem 1 (Binomial Theorem)**.** Suppose that $|x| < 1$, and that k is a real number. Then

$$(1+x)^k = 1 + \frac{k}{1!}x + \frac{k(k-1)}{2!}x^2 + \frac{k(k-1)k-2)}{3!}x^3 + \cdots.$$

Set $k = -1$, and set $x = -t$. We obtain the familiar "sum of an infinite geometric progression:"

$$\frac{1}{1-t} = 1 + t + t^2 + t^3 + \cdots. \qquad (1)$$

Now set $k = -1/2$ and $x = -t$. We obtain

$$\frac{1}{(1-t)^{1/2}} = 1 + \frac{1/2}{1!}t + \frac{(1/2)(3/2}{2!}t^2 + \frac{(1/2)(3/2)(5/2)}{3!}t^3 + \cdots.$$

After some manipulation we find that

$$\frac{1}{(1-t)^{1/2}} = b_0 + b_1t + b_2t^2 + b_3t^3 + \cdots. \qquad (2)$$

Note that

$$\frac{1}{(1-t)^{1/2}} \times \frac{1}{(1-t)^{1/2}} = \frac{1}{1-t}.$$

Now multiply the series expansion of $1/(1-t)^{1/2}$ (Equation 2)by itself in the "natural" way, as if this series expansion were a polynomial. The coefficient of t^n in this product is easily seen to be

$$b_0 b_n + b_1 b_{n-1} + b_2 b_{n-2} + \cdots + b_n b_0.$$

But the coefficient of t^n in this product is the same as the coefficient of t^n in the expansion of $1/(1-t)$, which is 1 for all n. This gives the desired result.

Comment. The argument given above is excessively informal. Beside the generalized Binomial Theorem, the argument assumes that products of infinite series of the type we used behave like products of polynomials. Rigorous proofs can be given for all the necessary facts, but it takes more than a little work.