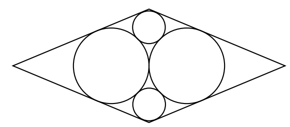
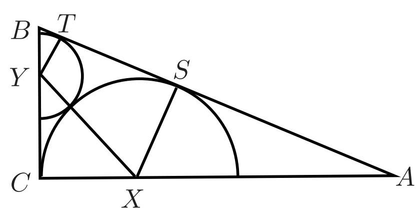
Solutions to May 2010 Problems

Problem 1. The figure below is a rhombus whose diagonals have lengths 72 and 30. Two congruent "large" circles are drawn, and then two congruent small circles, with tangencies as shown. What is the radius of a small circle?



Solution. For the solution, we get rid of the attractive symmetry—this may be a mistake–and concentrate on the "first-quadrant" part of the given diagram. Let $2D$ be the length of the longer (horizontal) diagonal, and let $2d$ be the length of the shorter vertical diagonal. (In our problem, we have $D = 36$ and $d = 15$.)

Let R be the radius of the larger circle, and r the radius of the smaller circle. Now look at the picture below. Note that we have labelled the centres of the semicircles—centres are almost always important in problems about circles. We have also labelled the points of tangency. And we have drawn line segments from the centres to the points of tangency, obtaining the diagram below.



Let c be the length of AB. Since $c^2 = d^2 + D^2$, we can take c as a known quantity. Finally, let u be the length of YC. Then $u^2 = (r + R)^2 - R^2$.

By the similarity of triangles ABC and AXS,

$$\frac{BC}{AB} = \frac{XS}{AX}.$$

Note that $BC = d$, $AB = c$, $XS = R$, $AX = D - R$,

$$\frac{d}{c} = \frac{R}{D - R}.$$

By the similarity of triangles ABC and YBT,

$$\frac{D}{c} = \frac{r}{d - u}$$

We know c and u in terms of the other variables, so we have ended up with two—unfortunately messy—equations in the "unknowns" r and R. By the first equation, we have $d(D - R) = cR$, and therefore $R = dD/(c+d)$. The second equation yields $u = (dD - rc)/D$. Since $u^2 = r^2 + 2rR$, we obtain the equation

$$r^2 + 2r\frac{dD}{c+d} = \left(\frac{dD - rc}{D}\right)^2.$$

Routine "cross-multiplication" yields

$$(c+d)D^2r^2 + 2dD^3r = (c+d)(d^2D^2 - 2dDcr + c^2r^2 = 0.$$

Simplify. Bring the stuff on the left to the right-hand side, and use the fact that $c^2 - D^2 = d^2$. The "r" term also simplifies pleasantly, to $-2dD(2c-d)(c+d)r$. After a while we obtain

$$dr^2 - 2D(2c-d)r + dD^2 = 0.$$

Solve (the bigger root is obviously too big). We obtain

$$r = \frac{D}{d}\left((2c-d) - 2\sqrt{c^2 - cd}\right).$$

Finally, we turn to our concrete problem, and put $D = 36$, $d = 15$. Then $c = 39$. And r turns out to be approximately 4.348238.

Comment. We could have gone "numerical" early, particularly since c turns out to be such a simple number. But why not go for generality?

Equations simplified too much! It probably indicates that I am missing a simpler way of doing the calculation.

Problem 2. The sequence a_0, a_1, a_2, and so on is defined by $a_0 = 2$ and $a_{n+1} = (2a_n + 1)/(a_n + 2)$ for $n \geq 0$. Find an explicit formula for a_n, and prove that the formula is correct.

Solution. It is useful to experiment. We can without much trouble calculate the first few a_i. We find that $a_1 = 5/4$, $a_2 = 14/13$, $a_3 = 41/40$. We can see the beginnings of a possible pattern: the numerator (at least so far) is 1 more than the denominator. The number a_0 also fits the pattern, after we note that $a_0 = 2/1$.

Here is a somewhat less obvious element of the pattern. Note that the sum of the numerator and denominator, for the first few terms, is 3, 9, 27, and 81. These are the powers of 3. So we may want to guess that the numerator of a_n is $(3^{n+1}+1)/2$ and the denominator is $(3^{n+1}-1)/2$. If we divide, and for simplicity cancel the 2s, we have the conjecture that

$$a_n = \frac{3^{n+1}+1}{3^{n+1}-1}.$$

We can prove the result by Mathematical Induction, a very important idea. But we will (sort of) sidestep doing a formal induction. Let $b_n = (3^{n+1}+1)/(3^{n+1}-1)$. We would like to show that $a_n = b_n$ for all n.

An easy calculation shows that $b_0 = 2$. We will show that

$$b_{n+1} = \frac{2b_n + 1}{b_n + 2}.$$

Like most identities, this is quite easy to prove. Note that

$$2b_n + 1 = 2\frac{3^{n+1}+1}{3^{n+1}-1} + 1.$$

Bringing the expression on the right to a common denominator, we find after not much calculation that

$$2b_{n+1}+1=\frac{2\cdot 3^{n+1}+3^{n+1}+1}{3^{n+1}-1}=\frac{3^{n+2}+1}{3^{n+1}-1}.$$

A similar calculation shows that

$$bn+1+2=\frac{3^{n+2}-1}{3^{n+1}-1}.$$

Finally, division gives the desired result.

We conclude that the sequence b_0, b_1, b_2 and so on begins in the same way as the sequence a_0, a_1, a_2 and so on. The sequence $\langle b_n\rangle$ also satisfies the same *recurrence relation* (that is, rule about what the "nest term" is) as the sequence $\langle a_n\rangle$. So the two sequences are the same.

Another Way. We worked too hard! Look at the somewhat more general problem with

$$a_{n+1}=\frac{ca_n+d}{da_n+c}$$

where c and d are real numbers (in our problem, $c=2$ and $d=1$). Leave a_0 unspecified. Let $a_i=p_i/q_i$. Then the equation above becomes, after a little manipulation,

$$\frac{p_{n+1}}{q_{n+1}}=\frac{cp_n+dq_n}{dp_n+cq_n}.$$

The above equation will be satisfied if

$$p_{n+1}=cp_n+dq_n \quad \text{and} \quad q_{n+1}=dp_n+cq_n.$$

The above system of equations may look more complicated than the original. But it has far more *symmetry,* and symmetry is almost always important.

In the system above, addition gives

$$p_{n+1}+q_{n+1}=(c+d)(p_n+q_n),$$

and subtraction gives

$$p_{n+1}-q_{n+1}=(c-d)(p_n-q_n).$$

Now suppose that we are given a_0. Express a_0 as a fraction $a_0=p_0/q_0$. We could for example take $p_0=a_0$ and $q_0=1$. Let $S=p_0+q_0$ and let $D=p_0-q_0$. The recurrence for p_i+q_i shows that in general

$$p_n+q_n=S(c+d)^n.$$

Similarly, the recurrence equation for p_i-q_i shows that

$$p_n-q_n=D(c-d)^n.$$

Finally, from the above explicit formulas for p_n+q_n and p_n-q_n, we can solve for p_n and q_n, and hence get an explicit formula for a_n.

Problem 3. (a) Find a simple expression for

$$1 \cdot \binom{n}{1} + 2 \cdot \binom{n}{2} + 3 \cdot \binom{n}{3} + \cdots + n \cdot \binom{n}{n}.$$

(b) Find a simple expression for

$$(2)(1)\binom{n}{2} + (3)(2)\binom{n}{3} + (4)(3)\binom{n}{4} + \cdots + (n)(n-1)\binom{n}{n}.$$

Solution. (a) The general term in the sum is $k\binom{n}{k}$. By the usual formula for $\binom{n}{k}$,

$$k\binom{n}{k} = k\frac{n!}{k!(n-k)!}.$$

Since $k! = k((k-1)!)$, it follows that $k/k! = 1/(k-1)!$. It is convenient to also use the fact that $n! = n((n-1)!)$. Thus

$$k\frac{n!}{k!(n-k)!} = n\frac{(n-1)!}{(k-1)!(n-k)!}.$$

Since $n - k = (n-1) - (k-1)$, we conclude that

$$k\binom{n}{k} = n\binom{n-1}{k-1}.$$

It follows that the sum we want to evaluate is equal to

$$n\left(\binom{n-1}{0} + \binom{n-1}{1} + \binom{n-1}{2} + \cdots + \binom{n-1}{n-1}\right).$$

Now we observe that in general

$$\binom{m}{0} + \binom{m}{1} + \binom{m}{2} + \cdots + \binom{m}{m} = 2^m.$$

There are many proofs. One can for example use the binomial expansion of $(1+x)^m$, and then set $x = 1$. But there is a neater way to do it. The sum on the left is the total number of ways of choosing 0, or 1, or 2, or ..., or m objects from a collection of m objects. This is the number of subsets of a set of m elements, and that is 2^m.

So we conclude that the sum of our problem is $n2^{n-1}$.

Another Way. The above argument was a little messy, a little ugly. We now give a much nicer *counting proof,* also known as a *combinatorial proof* (and sometimes more fancily called a *bijective proof.*

Imagine that we have a group of n people. We want to select a *chaired* committee, that is, a committee with a designated person on the committee who is called the chair. We put no restricitions on the size of the committee. It could even consist of a single person (who then is automatically the chair), or it could be a committee of the whole, that is, all n people, with a specified chair. Or the committee size could be anywhere in between.

We will count the number of such chaired committees in two ways. For the first way, imagine first deciding on the size k of the committee, choosing the committee, and then choosing one of the members to be the chair. A committee of k people can be chosen in $\binom{n}{k}$ ways. For each such choice of a committee, there are k ways to select the chair from within the committee. So there are $k \cdot \binom{n}{k}$ chaired committees of size k. If we allow all sizes k, the total number of chaired committees of all conceivable sizes is

$$1 \cdot \binom{n}{1} + 2 \cdot \binom{n}{2} + 3 \cdot \binom{n}{3} + \cdots + n \cdot \binom{n}{n},$$

precisely the sum we are interested in.

Now we count the number of chaired committees in a different way. First choose a person who will be the chair. This can be done in n ways. Then from the remaining $n-1$ people, choose any number (possibly 0) of people to join the chosen chair on the committee. So we want to choose a subset of a set of $n-1$ people. This can be done in 2^{n-1} ways. It follows that the number of chaired committees is $n2^{n-1}$.

We have counted the number of chaired committees (correctly!) in two ways, obtaining first our desired sum, and then $n2^{n-1}$. The counts are both correct, so they are equal, and we have our result.

(b) We show how to adapt the two approaches we used in part (a), in much less detail. To adapt the first approach, note that the general term is $k(k-1)\binom{n}{k}$. We showed how to "absorb" k. More precisely, we showed that

$$k\binom{n}{k} = n\binom{n-1}{k-1}.$$

Now in the same way, "absorb" $k-1$ in $\binom{n-1}{k-1}$. We get

$$k(k-1)\binom{n}{k} = n(k-1)\binom{n-1}{k-1} = n(n-1)\binom{n-2}{k-2}.$$

Thus our sum is equal to

$$n(n-1)\left(\binom{n-2}{0} + \binom{n-2}{1} + \binom{n-2}{2} + \cdots + \binom{n-2}{n-2}\right) = n(n-1)2^{n-2}.$$

Another Way. We now generalize the "chaired committee" approach. Every committee needs not only a chair, but an assistant chair to take over when the chair is sick. How many ways are there to choose a committee together with chair and assistant chair? We could first of all choose a chair (n ways). For every such choice, there are $n-1$ ways of choosing the assistant chair, for a total so far of $n(n-1)$ ways. And once we have chosen these two people, we can fill out the committee by choosing a subset (possibly empty) of the remaining $n-2$ people. For every choice of chair and assistant, there are 2^{n-2} ways of doing this last task, for a total of $n(n-1)2^{n-2}$

We count the number of ways to choose a committee with chair and assistant chair in another way. If the total number of people on the committee is to be k (where $k \geq 2$), first choose k people. This can be done in $\binom{n}{k}$ ways. For each such choice there are $k(k-1)$ ways to select a chair and assistant chair from within the group. So there are $k(k-1)\binom{n}{k}$ ways to choose a chaired and assistant chaired committee of k people. Now sum over all $k \geq 2$ to get all ways of getting a chaired and assistant chaired committee. But that sum is precisely the sum we wanted to evaluate. So our sum is equal to $n(n-1)2^{n-2}$.

Comment. Suppose that we want a simple expression for the sum

$$1^2\binom{n}{1} + 2^2\binom{n}{2} + 3^2\binom{n}{3} + 4^2\binom{n}{4} + \cdots + n^2\binom{n}{n}.$$

It is unpleasant to tackle this directly. However, note that $k^2 = k(k-1) + k$, so

$$k^2\binom{n}{k} = k(k-1)\binom{n}{k} + k\binom{n}{k}.$$

Now sum over all k. We get our sum of part (b) plus our sum of part (a). So the result is $n(n-1)2^{n-2} + n2^{n-1}$, which simplifies to $n(n+1)2^{n-2}$.

Problem 4. Is $\lfloor (1+\sqrt{2})^{999} \rfloor$ odd or is it even? (As usual, $\lfloor x \rfloor$ is the greatest integer which is $\le x$.)

Solution. Direct use of the calculator is hopeless. It will turn out that $(1+\sqrt{2})^{999}$ is very close to an enormous integer. No calculator will give the 999-th power to sufficient accuracy to answer our question. However, it might be interesting to see what happens if we replace 999 by 1, by 2, by 3, and so on for a while. It turns out that for $n = 1$, 3, 5, 7, $\lfloor (1+\sqrt{2})^n \rfloor$ is even, while for $n = 2$, 4, 6, and 8 it is odd. This might lead us to conjecture that this pattern continues. We will prove that it does.

Whenever a problem involves $a + b\sqrt{d}$, where a and b are rational, and d is a rational which is not the square of a rational, its friend (to be technically accurate, its *conjugate*) is nearly sure to be involved. Let n be a non-negative integer, and let

$$S(n) = (1+\sqrt{2})^n + (1-\sqrt{2})^n.$$

It is not hard to see that $S(n)$ is an integer, indeed an even integer. Here are a couple of ways of showing this. We could use the *Binomial Theorem,* which says that

$$(a+b)^n = \binom{n}{0}a^n b^0 + \binom{n}{1}a^{n-1}b + \binom{n}{2}a^{n-2}b^2 + \cdots + \binom{n}{n}a^0 b^n.$$

Letting $a = 1$ and $b = \sqrt{2}$, we find that

$$(1+\sqrt{2})^n = 1 + \binom{n}{1}\sqrt{2} + + \binom{n}{2}2 + \cdots + (\sqrt{2})^n.$$

Similarly, we find that

$$(1-\sqrt{2})^n = 1 - \binom{n}{1}\sqrt{2} + + \binom{n}{2}2 + \cdots + (-1)^n(\sqrt{2})^n.$$

In the expansion of $(1-\sqrt{2})^n$, alternate terms are negative. Note that these are precisely the terms that involve *odd* powers of $\sqrt{n}$. So when we add the expansions of $(1+\sqrt{2})^n$ and $(1-\sqrt{2})^n$, we get

$$2 + 2\binom{n}{2}(\sqrt{2})^2 + 2\binom{n}{4}(\sqrt{2})^4 + \cdots .$$

But even powers of $\sqrt{2}$ are plainly integers, so every number in the sum above is an even integer, and therefore $S(n)$ is an even integer.

In our particular case, we conclude that

$$(1+\sqrt{2})^{999}+(1-\sqrt{2})^{999}$$

is an even integer. But $(1-\sqrt{2})^{999}$ is negative (and extremely close to 0). So $(1+\sqrt{2})^{999}$ is the even integer $A(999)$ plus something positive but very close to 0. Thus $\lfloor(1+\sqrt{2})^{999}\rfloor$ is even. If we replace 999 by 1000, then $(1-\sqrt{2})^{1000}$ is an even integer minus something positive but close to 0. Thus $\lfloor(1+\sqrt{2})^{1000}\rfloor$ is odd.

Another Way. The next approach is more conceptual, and in the long run more useful. Let $x = a+b\sqrt{d}$, where a and b are rational, or if you prefer, integers, and d is a rational which is not the square of a rational. The *conjugate* of x, often denoted by $\overline{x}$, is defined by

$$\overline{x} = a - b\sqrt{d}.$$

A special (and specially important) case is the case $d = -1$, where in fact we do not even insist that a and b be rational. Write as usual i for one of the two square roots of -1. (Perhaps I should not say "as usual": electrical engineers use j instead of i.) Then $a - bi$ is called the *complex conjugate* of $a + bi$.

It is easy to check that in general $\overline{x+y} = \overline{x}+\overline{y}$, and that $\overline{xy} = \overline{xy}$. And, trivially, $\overline{\overline{x}} = x$. It is also easy to check that $\overline{x} = x$ precisely if x is rational, that is, precisely if "b" is 0.

Using the above facts, we can see that

$$\overline{(a+b\sqrt{d})^n} = (\overline{a+b\sqrt{d}})^n = (a-b\sqrt{d})^n$$

(with corresponding result for $\overline{(a-b\sqrt{d})^n}$) and therefore

$$\overline{(a+b\sqrt{d})^n+(a-b\sqrt{d})^n} = (a-b\sqrt{d})^n+(a+b\sqrt{d})^n,$$

from which we can conclude that if a, b, and d are integers, then $(a+b\sqrt{d})^n+(a-b\sqrt{d})^n$ is always an integer. Nice, but not quite enough. We would like to show that it is an even integer. So we modify our approach slightly.

In general, if a, b, p, q, and d are integers (with d not a perfect square), then $(a+b\sqrt{d})(p+q\sqrt{d})$ is of the form $r+s\sqrt{d}$ where r and s are integers. It follows that $(a+b\sqrt{d})^n$ is of the shape $A_b + B_n\sqrt{d}$, where A_n and B_n are integers. And its conjugate $(a-b\sqrt{d})^n$ is then $A_n - B_n\sqrt{d}$, giving a sum of $2A_n$, twice an integer.