Solutions to May 2011 Problems

Problem 1. Given six points in the *interior* of a circle of radius 1, show that the distance between two of these points is less than 1. This is obvious, maybe, but if it is one should be able to prove it.

Solution. If one of the points is the center O of the circle, then the distance from this point to each of the other 5 is less than 1. Now let A be one of the points, and draw the radial half-line from O in the direction of A. Imagine rotating this half-line counterclockwise about O. Let B, C, D, E, and F be, *in order,* the 5 remaining points that we meet as we rotate the half-line.

Now consider the 6 angles AOB, BOC, COD, DOE, EOF, FOA (some of these angles may be 0). The sum of these angles is $360°$, so one of these angles, without loss of generality AOB, is $\leq 60°$. We will show that $AB < 1$.

Look at $\triangle AOB$. Since $\angle AOB \leq 60°$, at least one of the remaining angles of the triangle is $\geq \angle AOB$, and therefore one at least of the sides OA and OB is $\geq AB$. But each of OA and OB is < 1, so $AB < 1$.

Problem 2. Find (with proof) all functions f from the reals to the reals such that

$$x(f(y))^2 + y(f(x))^2 = (x+y)f(x)f(y)$$

for all reals x, y.

Solution. The equation can be rewritten as

$$xf(y)(f(y) - f(x)) = yf(x)(f(y) - f(x)). \tag{1}$$

This equation is satisfied if $f(x) = f(y)$ for all x, y. That gives the solutions $f(x) = k$ where k is any constant. Now we look for non-constant solutions.

If f is such a solution, set $x = 0$, and $y = b$, where b is any real number such that $f(b) \neq f(0)$ (there is such a b since f is not constant). Then in Equation (1), the left-hand side is 0, and in the right-hand side, $b(f(b) - f(0)) \neq 0$. It follows that $f(0) = 0$.

Suppose that $f(1) = k$. Set $x = 1$, and suppose that $f(y) \neq k$. Then in Equation (1), the terms $f(y) - f(x)$ can be cancelled, and we obtain $f(y) = ky$.

Let's take stock of what we have so far. If $k = 0$, then $f(y) = 0$ for all y, which is one of the constant solutions described earlier. So suppose that $k \neq 0$. Suppose there is a non-zero a such that $f(a) \neq k$, and suppose that b is a real number such that $f(b) = k$. Set $x = a$ and $y = b$. Then from Equation (1), we obtain

$$af(b) = bf(a)$$

since the $f(b) - f(a)$ terms cancel. But $f(a) = ka$ and $f(b) = k$, and therefore $ka = bka$, so $b = 1$. We conclude that if for some non-zero a we have $f(a) \neq k$, then the only value of y such that $f(y) = k$ is given by $y = 1$. Thus if for some non-zero a we have $f(a) \neq k$, then $f(x) = kx$ for all x.

So the only possibilities are (i) $f(x)$ is constant; (ii) $f(x) = kx$ for some constant k; (iii) $f(x) = k$ for some constant k and all $x \neq 0$, and $f(0) = 0$. It is easy to verify that our equation is satisfied in all three of these cases.

Problem 3. Let a, b, and c be the roots of the cubic equation $x^3 + 3x^2 - 1 = 0$. Write down a cubic polynomial whose roots are a^2, b^2, and c^2. Finding a, b, and c may be difficult. Can we solve the problem without doing that?

Solution. The polynomial $P(x) = (x - p)(x - q)(x - r)$ has roots p, q, and r. Expand. We obtain

$$P(x) = x^3 - (p + q + r)x^2 + (pq + qr + rp)x - pqr.$$

Thus $a + b + c = -3$, $ab + bc + ca = 0$ and $abc = 1$. We need to calculate the coefficients of the polynomial

$$x^3 - (a^2 + b^2 + c^2)x^2 + (a^2b^2 + b^2c^2 + c^2a^2)x - a^2b^2c^2.$$

The easiest coefficient to find is $a^2b^2c^2$. It is $(abc)^2$, which is 1. Now we calculate $a^2 + b^2 + c^2$. Note that

$$a^2 + b^2 + c^2 = (a + b + c)^2 - 2(ab + bc + ca).$$

It follows that $a^2 + b^2 + c^2 = (-3)^2 - 2(0) = 9$. Finally, we find $a^2b^2 + b^2c^2 + c^2a^2$. It seems sensible to square $ab + bc + ca$:

$$(ab + bc + ca)^2 = a^2b^2 + b^2c^2 + c^2a^2 = a^2b^2 + b^2c^2 + c^2a^2 + 2(ab^2c + bc^2a + ca^2b).$$

But $ab^2c + bc^2a + ca^2b = abc = abc(b + c + a) = -3$, so $a^2b^2 + b^2c^2 + c^2a^2 = 6$. It follows that the cubic polynomial $x^3 - 9x^2 + 6x - 1$ has the right roots.

Problem 4. A set of integers is called *double-free* if for any x in the set, $2x$ is not in the set. (a) What is the largest possible size of a double-free subset of $\{1, 2, 3, \ldots, 2011\}$? (The answer can be found without working too hard.) (b) How many double-free subsets of this size are there?

Solution. (a) We start with a natural but somewhat painful approach. Look first at the powers of 2 less than or equal to 2011, namely 1, 2, 4, ..., 1024. There are 11 such powers. To make a largest double-free set, we select 1, 4, 16, and so on to 1024, a total of 6. Then look at $3 \cdot 1$, $3 \cdot 2$, $3 \cdot 4$, and so on. The biggest number in this list less than or equal to 2011 is $3 \cdot 512$, so there are 10 of them. Take every second one, *either* either $3 \cdot 1$, $3 \cdot 4$, and so on, or $3 \cdot 2$, $3 \cdot 8$, and so on. Whichever we grab, we get a total of 5.

Next comes 5 times a power of 2. The powers of 2 go all the way to the smallest power of 2 that is less than $\lfloor 2011/5 \rfloor$, so to 256. There are 9 of them, so we grab 1, 4, ..., 256, a total of 5. Next look at 7 times a power of 2. Again the largest power of 2 is 256, so we get 5 more.

Then we deal with 9, 11, 13, and 15, a total of 4 odd numbers. This time the largest power of 2 is 128, there are 8 of them, and we can grab *either* 1, 4, ... or 2, 8, ... Whichever we grab, we get a total of 16 numbers. Then come 17, 19, ..., 31, a total of 8 odd numbers. The largest power of 2 is 64, so we use 1, 4, ..., 64, for a total of 32.

Then come 33, 35, ..., 61, a total of 15 odd numbers. In each case the largest power of 2 is 32, so in each case we can take *either* 1, 4, 16 or 2, 8, 32. We get a total of 45. Then come 63, 65, ..., 125, a total of 32 odd numbers. The largest power of 2 is 16, we take 1, 4, 16, a total of 96.

Then come 127, 129, ..., 251, a total of 63 odd numbers. The largest power of 2 is 8, we take *either* 1, 4 or 2, 8, for a total of 126. Then look at 253, 255, ..., 501, a total of 125 odd numbers.

The largest power of 2 is 4, so we take 1, 4, for a total of 250. Then look at 503, 505, ..., 1005, a total of 252 odd numbers. The largest power of 2 is 2, so we use *either* 1 or 2, a total of 252. Finally, we take 1007 to 2011, 503 odd numbers, for a total of 503.

Whew! Add up. If everything went well, we get a grand total of 1341.

Another Way. Perhaps we could have started in a more efficient way, or perhaps the hard slogging of the first solution was necessary in order to find an improvement. We will use all the odd numbers up to 2011, a total of 1006. Then we take the numbers of the form 4 times an odd number. Since $2011/4 \approx 502.7$, we use the odd numbers 1 to 501, a total of 250. Then we take the numbers of the form 16 times an odd number. Since $2011/16 \approx 125.6$, we use the odd numbers 1 to 125, 63 of them. Next we look at 64, 256, 1024. The numbers we get are 16, 4, and 1. Add up. The sum is 1341, amazingly enough the same as with the first complicated approach.

Another Way. We describe a slightly more complicated version of the preceding solution, which introduces an often useful idea, called the *Principle of Inclusion-Exclusion* (PIE). Let us as a first estimate decide to put in all 2011 numbers from 1 to 2011. That is no good, we should throw out all the even numbers, of which there are $\lfloor 2011/2 \rfloor$. We have thrown out too many, so we should put back the $\lfloor 2011/4 \rfloor$ multiples of 4. But this puts back too many, we should remove the $\lfloor 2011/8 \rfloor$ multiples of 8. This removes too many, ... So the desired number is $\lfloor 2011/1 \rfloor - \lfloor 2011/2 \rfloor + \lfloor 2011/4 \rfloor - \lfloor 2011/8 \rfloor + \cdots$. Now calculate.

(b) We go back to the first, complicated approach. With 3, we had 2 choices as to which numbers to select. We also had 2 choices for each of 9, 11, 13, 15, and also for 33, 35, ..., 61, also for 127, 129, ..., 251, also for 503, 505, ..., 1005. So we had 2 choices in $1 + 4 + 15 + 63 + 252$, that is, 335 cases. It follows that the number of double-free subsets of maximum size is 2^{335}. Lots!