5 Bifurcations — summary

Definition. Bifurcation — A forking, or division into two branches.
(from Webster's Revised Unabridged Dictionary (1913))

Definition. A 1-parameter family of functions, F_λ undergoes a *saddle-node bifurcation* or *tangent bifurcation* at parameter value λ_0 if $\exists$ an open interval I and an $\varepsilon > 0$ such that

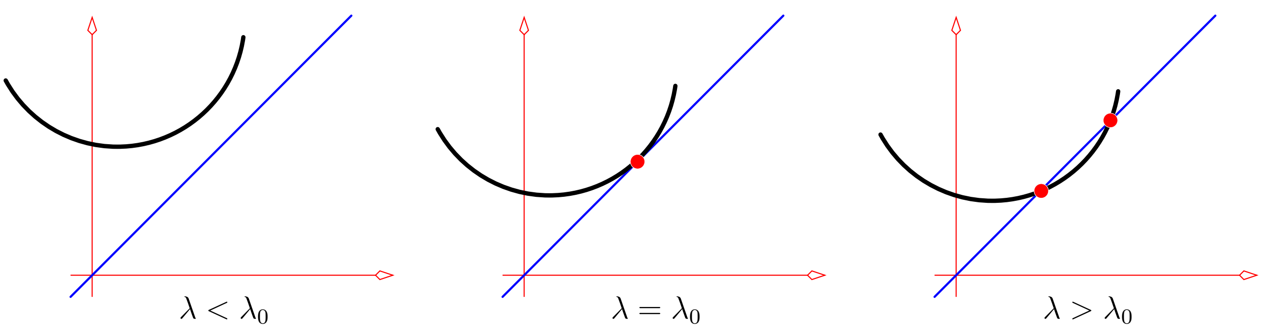
1. if $\lambda_0 - \varepsilon < \lambda < \lambda_0$ then F_λ has no fixed points in I.
2. if $\lambda = \lambda_0$ then F_λ has a unique fixed point in I which is neutral.
3. if $\lambda_0 > \lambda > \lambda_0 + \varepsilon$ then F_λ has two fixed points in I, one attracting and one repelling.

Note that the inequalities need not be this way around, we could equally as well have:

$\bar{1}$ if $\lambda_0 > \lambda > \lambda_0 + \varepsilon$ then F_λ has no fixed points in I.

$\bar{3}$ if $\lambda_0 - \varepsilon < \lambda < \lambda_0$ then F_λ has two fixed points in I, one attracting and one repelling.

Also note that periodic points also undergo saddle node bifurcations; if the point has period n, then simply replace F_λ in the above definition with F_λ^n.



Typically a saddle node bifurcation occurs when a function (such as $x^2 + x$) pushes down through the line $y = x$ and its point of contact is tangent (and so has $f'(x) = 1$). Thus while the function lies above $y = x$ there are no fixed points (in the region), it has a single neutral fixed point when the curves touch, and finally two fixed points appear when the curve cuts through $y = x$.

Definition. A 1-parameter family of functions, F_λ undergoes a *period doubling bifurcation* at parameter value λ_0 if $\exists$ an open interval I and an $\varepsilon > 0$ such that

1. if $\lambda_0 - \varepsilon < \lambda < \lambda_0 + \varepsilon$ then F_λ has a unique fixed point, p_λ in I.

2. if $\lambda_0 - \varepsilon < \lambda < \lambda_0$ then F_λ has no 2-cycles in I and the fixed point, p_λ, is attracting.

3. if $\lambda_0 > \lambda > \lambda_0 + \varepsilon$ then F_λ has a unique 2-cycle, q_λ^1, q_λ^2 in I. This 2-cycle is attracting while the fixed point, p_λ is repelling.

4. As $\lambda \to \lambda_0^+$ we have $q_\lambda^1, q_\lambda^2 \to p_\lambda$

As was the case for saddle node bifurcations, it is possible that the inequalities are the other way around — decreasing λ gives rise to the attracting 2-cycle, while increasing λ gives the attracting fixed point and no 2-cycle.

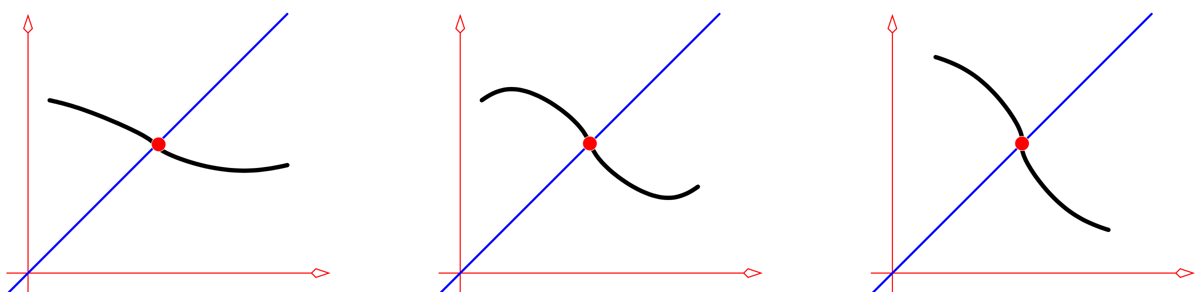
Notes:

1. In saddle node bifurcation the neutral fixed point splits into two fixed points, while in period double bifurcation the 2-cycle is born from the fixed point, but the fixed point continues to exist.

2. The attracting fixed point in period doubling bifurcation becomes repelling at the same time as the attracting 2-cycle is born at the fixed point.

3. Cycles may also undergo a period doubling bifurcation; an n-cycle gives birth to a $2n$-cycle.

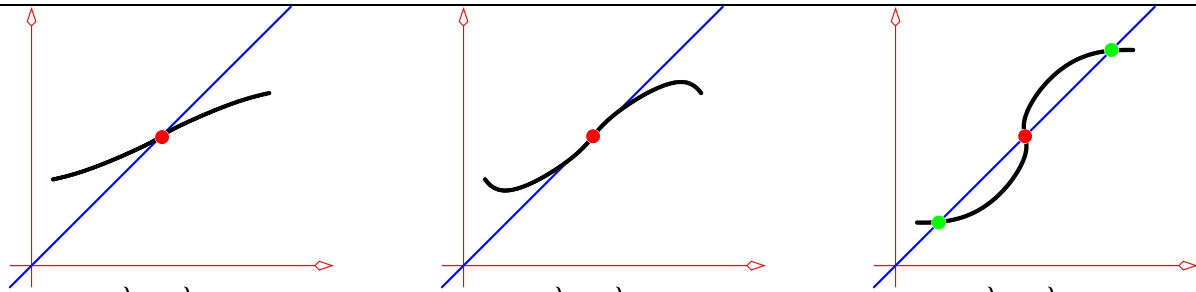
Typically we find that a period doubling bifurcation occurs when the graph of $f(x)$ is perpendicular to $y = x$ at the fixed point — *i.e.* $f'(p_\lambda) = -1$.



$\lambda < \lambda_0$ $\quad$ $\lambda = \lambda_0$ $\quad$ $\lambda > \lambda_0$

Since the fixed point is attracting for smaller λ and repelling for larger λ we must have graphs that look something like those above.

Since we expect that a 2-cycle has been born as λ passes through λ_0 it makes sense to examine a graph of $f^2(x)$. We first note that $(f^2)'(p_\lambda) = (f'(p_\lambda))^2 = 1$. Hence for $\lambda < \lambda_0$ we expect $(f^2)' < 1$ and for $\lambda > \lambda_0$ we expect $(f^2)' > 1$. This produces graphs looking something like:



$\lambda < \lambda_0$ $\lambda = \lambda_0$ $\lambda > \lambda_0$

This shows how an attracting 2-cycle is born from a repelling fixed point as the function $f^2(x)$ twists through the line $y = x$.