3 Fixed points — summary

Theorem (Fixed point existence).
Let $F : [a, b] \mapsto [a, b]$ be a continuous function. Then F has at least one fixed point in $[a, b]$.

Definition. Let x_* be a fixed point of a differentiable function $f(x)$.

- If $|f'(x_*)| < 1$ then we call x_* an *attracting* fixed point.
- If $|f'(x_*)| > 1$ then we call x_* a *repelling* fixed point.
- If $|f'(x_*)| = 1$ then we call x_* a *neutral* fixed point.

Theorem (Attracting fixed point theorem).
Let x_* be an attracting fixed point of a differentiable function $f(x)$ (so that $|f'(x_*)| < 1$). Then there exists an interval, I, which contains x_* as an interior point for which the following is true:

- if $x \in I$ then $f^n(x) \in I$ for all $n > 0$, and
- for all $x \in I$, $f^n(x) \longrightarrow x_*$ as $n \to \infty$.

Theorem (Repelling fixed point theorem).
Let x_* be a repelling fixed point of a differentiable function $f(x)$ (so that $|f'(x_*)| > 1$). Then there exists an interval, I, which contains x_* as an interior point for which the following is true:

- if $x \in I$ and $x \neq x_*$ then $\exists n > 0$ such that $f^n(x) \notin I$.

Note: This tells us that a point near a repelling fixed point will be pushed away after some n iterations. However it does not tell us what will happen subsequently — it could re-enter the interval — this depends on the global properties of the function, rather than on the local properties close to the fixed point.

Neutral fixed points: Neutral fixed points can display quite different behaviour:

- they can be *weakly attracting* (nearby points converge slowly to the fixed point) — such as the point $x = 0$ for $f(x) = x - x^3$.
- they can be *weakly repelling* (nearby points are slowly pushed away from the fixed point) — such as the point $x = 0$ for $f(x) = x + x^3$.
- they can be neither attracting nor repelling — such as $x = 0$ for the $f(x) = x - x^2$ — which is repelling to the left and attracting to the right.

Theorem (Chain rule along a cycle).
Let $\{x_0, x_1, \ldots, x_{n-1}\}$ be an n-cycle of the differentiable function f — with $x_i = f^i(x_0)$. Then

$$(f^n)'(x_0) = f'(x_0)f'(x_1)\ldots f'(x_{n-1})$$

Since the x_i lie on a cycle, $(f^n)'(x_i) = (f^n)'(x_0)$ for all i.

Using this we similarly define attracting, repelling and neutral periodic points, by noting that a periodic point of the function $f(x)$, with period n is a fixed point of the function $f^n(x)$.