3 Problem Set 3 — Fixed and periodic points

1. Find and classify the fixed points of the following functions:

   (a) $F(x) = x(1-x)$

   (b) $F(x) = 3x(1-x$

   (c) $F(x) = \frac{7}{2}x(1-x)$

   (d) $F(x) = x^4 - 4x^2 + 2$

   (e) $F(x) = \frac{\pi}{2}\sin x$

   (f) $F(x) = \arctan x$

   (g) $F(x) = x^{-2}$

2. The point $x = 0$ lies on a periodic orbit for each of the following functions. Classify the orbit.

   (a) $F(x) = 1 - x^2$

   (b) $F(x) = \frac{\pi}{2}\cos x$

   (c) $F(x) = -\frac{4}{\pi}\arctan(1+x)$

   (d) $F(x) = |x-2| - 1$

   (e) $F(x) = \begin{cases} x+1 & x \leq \frac{7}{2} \\ 2x-8 & x > \frac{7}{2} \end{cases}$

3. The doubling function is defined as

$$D(x) = \begin{cases} 2x & 0 \leq x < \frac{1}{2} \\ 2x-1 & \frac{1}{2} \leq x < 1 \end{cases}.$$

   Suppose that a point x_0 lies on a cycle of prime period n. By evaluating $(D^n)'$ (or otherwise) classify the orbit.

4. Each of the following functions has a neutral fixed point. Find the fixed point and determine whether it is weakly attracting, weakly repelling or neither. Plot an accurate graph and use graphical analysis to do so.

   (a) $F(x) = 1/x$

   (b) $F(x) = \tan(x)$

   (c) $F(x) = x + x^2$

(d) $F(x) = e^{x-1}$

(e) $F(x) = \log|x-1|$

5. Suppose $F(x)$ has a neutral fixed point at x_0, with $F'(x_0) = 1$.

   (a) Suppose that $F''(x_0) > 0$. Is x_0 weakly attracting, repelling or neither?

   (b) Suppose that $F''(x_0) < 0$. Is x_0 weakly attracting, repelling or neither?

   Use graphical analysis and the concavity of F near x_0 to support your answer.

6. Suppose $F(x)$ has a neutral fixed point at x_0. Further $F'(x_0) = 1$ and $F''(x_0) = 0$.

   (a) Show that if $F'''(x_0) > 0$ then x_0 is weakly repelling.

   (b) Show that if $F'''(x_0) < 0$ then x_0 is weakly attracting.

   Again use graphical analysis and the concavity of F to support your answer.