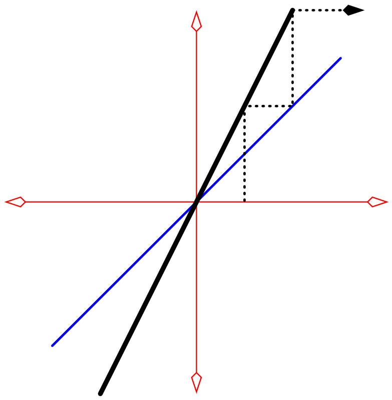
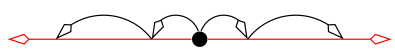
2 Problem Set 2 — Graphical Analysis

1. Use graphical analysis to describe all orbits of the functions below. Also draw their phase portraits.

   (a) $F(x) = 2x$

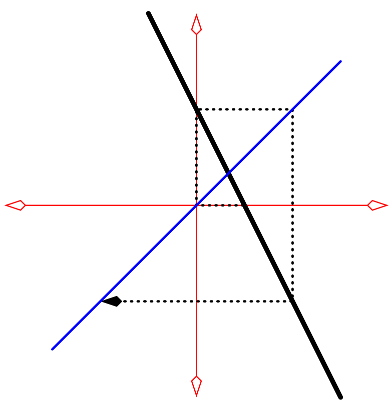
   - There is only one fixed point at $x = 0$ and it is repelling:

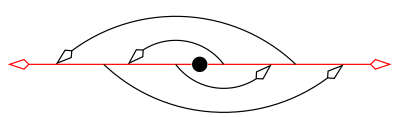




   (b) $F(x) = 1 - 2x$

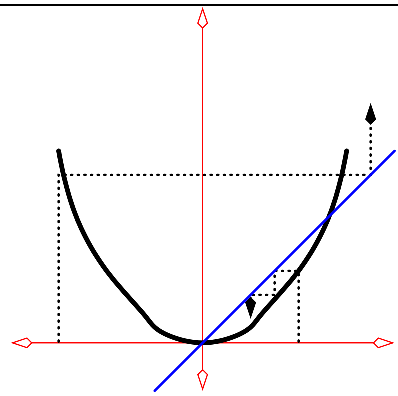
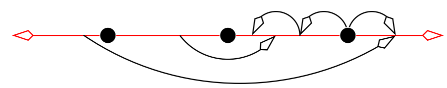
   - There is only one fixed point at $x = 1/3$ and it is repelling.





   (c) $F(x) = x^2$

   - There are two fixed points. $x = 0$ is an attracting fixed point, and $x = 1$ is a repelling fixed point. Also $x = -1$ is an eventually fixed point.

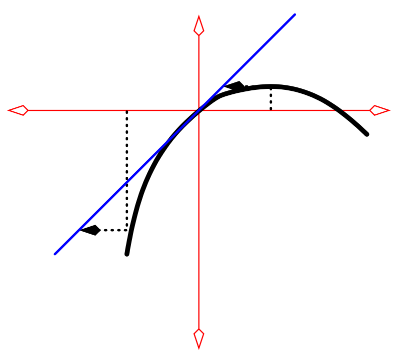
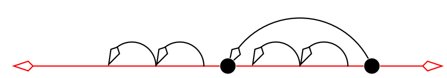



We see that if $0 < x_0 < 1$ then $x_n \to 0$. While if $x_0 > 1$ then $x_n \to \infty$. If $x_0 < 0$ but $x_0 > -1$ then $0 < x_1 < 1$ and so $x_n \to 0$. Finally if $x_0 < -1$ then $x_1 > 1$ and $x_n \to \infty$. Hence

$$\lim_{n\to\infty} F^n(x_0) = \begin{cases} 0 & \text{if } |x_0| < 1 \\ 1 & \text{if } |x_0| = 1 \\ \infty & \text{if } |x_0| > 1 \end{cases}$$

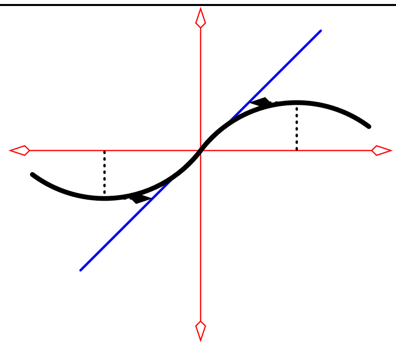
(d) $F(x) = x - x^2$

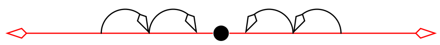
- $F(x)$ has a single fixed point at $x = 0$. We also see that $x = 1$ is eventually fixed at $x = 0$.

(e) $F(x) = \sin(x)$

- Since $F : \mathbb{R} \mapsto [-1, 1]$, no matter which $x_0 \in \mathbb{R}$ we choose, $x_1 \in [-1, 1]$, so we only need to consider points within this interval. It is clear that there is only a single fixed point at $x = 0$ and that points converge to it. Numerical experiments show that the convergence is very slow — it is a neutral fixed point (which is weakly attracting).

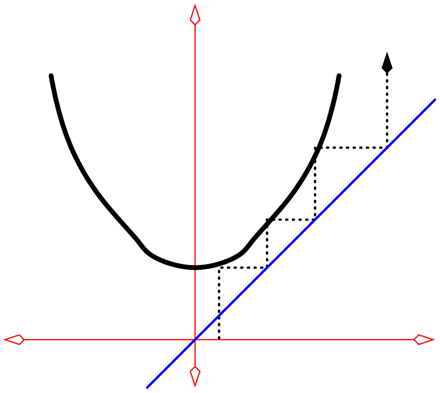




2. Use graphical analysis to find all the points whose orbits tend to infinity, *i.e.* $\{x_0 \mid F^n(x_0) \to \pm\infty\}$, for the following functions:

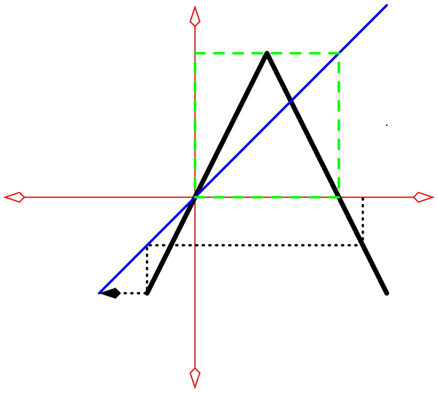
   (a) $F(x) = x^2 + 1$

   - From a graph of the function we see that there are no fixed points and that all iterates tend to infinity. *i.e.* $\forall x \in \mathbb{R}, F_n(x) \to \infty$.



   (b) $F(x) = \begin{cases} 2x & 0 \le x \le 1/2 \\ 2 - 2x & 1/2 < x \le 1 \end{cases}$

   - $F(x)$ maps the interval $[0,1]$ to itself and so all orbits that start in $[0,1]$ will stay within $[0,1]$. On the other hand, if $x_0 < 0$ then 0 is a repelling fixed point and $x_n \to \infty$. Similarly if $x_0 > 1$ then $x_1 < 0$ and $x_n \to \infty$.



3. Completely analyse the orbits of the following functions:

   (a) $F(x) = \frac{1}{2}x - 2$

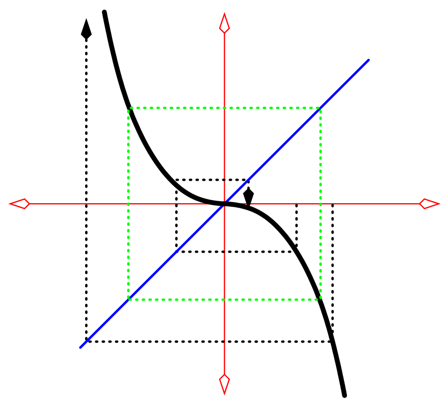
   - There is only a single fixed point at $x = -4$ and it is attracting. $\forall x \in \mathbb{R}, F^n(x) \to -4$.

   (b) $F(x) = |x|$

   - For $x \geq 0$, $|x| = x$, so all $x \geq 0$ are fixed points. If $x < 0$ then $|x| = -x > 0$, and so all $x < 0$ are eventually fixed.

   (c) $F(x) = -x^5$

   - This function has only a single real fixed point at $x = 0$. A plot of $F(x)$ shows that points close to 0 have orbits that are attracted to 0. However points far from 0 will be repelled away from the origin towards $\pm\infty$. This is because of the presence of a repelling 2-cycle at $\{-1, 1\}$. Hence if $|x_0| < 1$ then $x_n \to 0$, while if $|x| > 1$ then $x_n \to \pm\infty$.

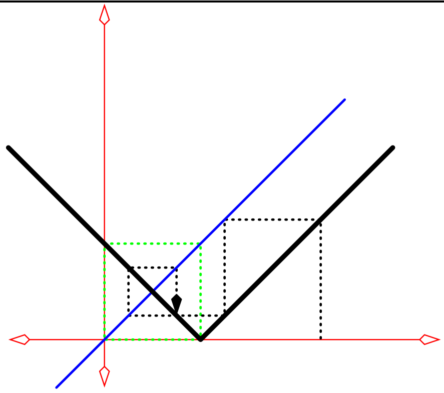


   (d) $F(x) = e^x$

   - A plot of F shows that this function has no fixed point and that $F^n(x) \to \infty$ for all $x \in \mathbb{R}$.

4. Analyse the orbits of the function $F(x) = |x - 2|$. Draw different types of orbits in different colours. You will be able to find fixed points, eventually fixed points, periodic points and eventually periodic points.

   - We see that at $x = 1$ there is the only fixed point. Further we can see from a plot of F that there is a two-cycle at $\{0, 2\}$.



If $x_0 < 0$ then $x_1 > 0$ — so let us look at orbits of points > 0. If $x_0 > 2$ then $x_1 = x_0 - 2$. Thus all points in $\mathbb{R}$ will be eventually attracted to the interval $[0, 2]$. All odd integers are eventually fixed at $x = 1$, while all even integers are eventually periodic at the 2-cycle $\{0, 2\}$. All other points in the interval $(0, 2)$ are periodic with period 2:

$$F(F(x)) = ||x-2|-2| = |(2-x)-2| = x \qquad \text{for } 0 < x < 2.$$

Hence all points in $\mathbb{R}$ are eventually periodic with period 2, excepting the odd integers which are eventually fixed.

5. Let $F(x) = x^2 - \frac{6}{5}$. Find the fixed point(s) of F. Using the fixed point(s) (or otherwise) find the cycle of prime period 2.

   - The fixed points of $F(x)$ are also the fixed points of $F(F(x))$. Hence the solutions of $F(x) - x = 0$ are also solutions of $F(F(x)) - x = 0$. The fixed points are solutions of $x^2 - x - \frac{6}{5} = 0$, which are $\frac{1}{2} \pm \frac{\sqrt{145}}{10}$. If we now turn to the solutions of $F(F(x)) = x$, then we already know that $\frac{1}{2} \pm \frac{\sqrt{145}}{10}$ are solutions, and so $x^2 - x - \frac{6}{5} = 0$ is a factor of $F(F(x)) - x$.

$$\begin{aligned} F(F(x)) - x &= (x^2 - \frac{6}{5})^2 - \frac{6}{5} - x \\ &= x^4 - \frac{12}{5}x^2 - x + \frac{6}{25} \\ &= (x^2 - x - \frac{6}{5})(x^2 + x - \frac{1}{5}) \end{aligned}$$

   The first factor gives the fixed points, and the second will give the 2-cycle:

$$p_{\pm} = -\frac{1}{2} \pm \frac{3\sqrt{5}}{10}$$

   It is easy to check that $F(p_{\pm}) = p_{\mp}$.

6. Let $F(x) = ax + b$. Answer the following questions about the dynamics of F for various values of a and b:

(a) Find the fixed points of F.

- Solving $ax + b = x$, gives $x_* = \frac{b}{1-a}$.

(b) For what values of a and b does F have no fixed points?

- If $a \neq 1$ then the system will always have a fixed point. On the other hand, if $a = 1$ then the fixed point equation becomes $b = 0$. Hence if $a = 1, b \neq 0$ then there are no fixed points. While if $a = 1$ and $b = 0$ then all points are fixed.

(c) For what values of a and b does F have infinitely many fixed points?

- See previous.

(d) For which values of a and b does F have *exactly* one fixed point?

- If $a \neq 1$ then the system will have exactly one fixed point at $x = \frac{b}{1-a}$.

(e) If F has exactly one fixed point and $|a| < 1$, what is the behaviour of all orbits under F? Use graphical analysis.

- This is much the same as some of the linear examples in question 1 — all orbits tend to the fixed point.

(f) Similarly, if $|a| > 1$ what is the behaviour of all orbits under F?

- Again this is much as question 1 — all orbits tend to infinity (except the fixed point).

(g) If $a = 1$ describe the orbits of F for $b < 0$, $b = 0$ and $b > 0$?

- If $b = 0$ then all points are fixed. On the other hand, if $b < 0$, then $F(x) < x$ and all orbits tend to $-\infty$. Similarly if $b > 0$ then $F(x) > x$ and all orbits tend to $+\infty$.

(h) Similarly, if $a = -1$ describe the orbits of F for $b < 0$, $b = 0$ and $b > 0$?

- We find a fixed point at $x = b/2$, and all other points form 2-cycles:

$$F(F(x)) = -(-x + b) + b = x.$$