3 Problem Set 3 — Fixed and periodic points

1. Find and classify the fixed points of the following functions:

   (a) $F(x) = x(1-x)$
      - Single neutral fp at $x = 0$.

   (b) $F(x) = 3x(1-x)$
      - Repelling fp at $x = 0$ and neutral fp at $x = 2/3$.

   (c) $F(x) = \frac{7}{2}x(1-x)$
      - Repelling fp at $x = 0$ and $x = 5/7$.

   (d) $F(x) = x^4 - 4x^2 + 2$
      - Has 4 repelling fixed points: $x = 2, -1, \frac{1}{2} \pm \frac{\sqrt{5}}{2}$.

   (e) $F(x) = \frac{\pi}{2} \sin x$
      - Has 3 fp. $x = 0$ is a repelling fp. $x = \pm\frac{\pi}{2}$ are attracting.

   (f) $F(x) = \arctan x$
      - Single neutral fp at $x = 0$.

   (g) $F(x) = x^{-2}$
      - Single repelling fp at $x = 1$.

2. The point $x = 0$ lies on a periodic orbit for each of the following functions. Classify the orbit.

   (a) $F(x) = 1 - x^2$
      - Orbit is $\{0, 1\}$. It is attracting.

   (b) $F(x) = \frac{\pi}{2} \cos x$
      - Orbit is $\{0, \pi/2\}$. It is attracting.

   (c) $F(x) = -\frac{4}{\pi} \arctan(1 + x)$
      - Orbit is $\{0, -1\}$. It is attracting.

   (d) $F(x) = |x - 2| - 1$
      - Orbit is $\{0, 1\}$. It is neutral.

   (e) $F(x) = \begin{cases} x + 1 & x \leq \frac{7}{2} \\ 2x - 8 & x > \frac{7}{2} \end{cases}$
      - Orbit is $\{0, 1, 2, 3, 4\}$. It is repelling.

3. The doubling function is defined as

$$D(x) = \begin{cases} 2x & 0 \le x < \frac{1}{2} \\ 2x - 1 & \frac{1}{2} \le x < 1 \end{cases}.$$

Suppose that a point x_0 lies on a cycle of prime period n. By evaluating $(D^n)'$ (or otherwise) classify the orbit.

- We first note that $x_0 \neq 1/2$. This is because $1/2$ is eventually fixed. Similarly it is not possible that $x_k = 1/2$ for any $k > 0$ since it implies that x_0 is eventually fixed. Now if x_0 has period n, it implies that $D^n(x_0) = x_0$. To determine the nature of the periodic point we take the derivative:

$$(D^n)'(x_0) = \prod_{k=0}^{n-1} D'(x_k) = 2^n,$$

since all points except $x = 1/2$ have gradient 2. Hence all periodic points are repelling.

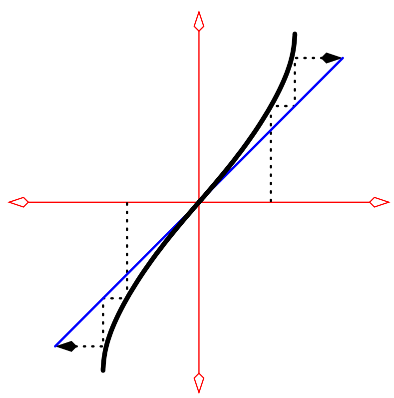
4. Each of the following functions has a neutral fixed point. Find the fixed point and determine whether it is weakly attracting, weakly repelling or neither. Plot an accurate graph and use graphical analysis to do so.

(a) $F(x) = 1/x$

- Since $F(F(x)) = x$ it follows that all points (except $x = 1$ which is fixed) are periodic with period 2. No graph required.

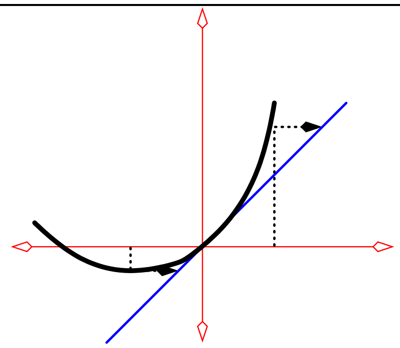
(b) $F(x) = \tan(x)$

- A careful plot of $\tan(x)$ shows that the neutral fixed point at $x = 0$ is in weakly repelling:



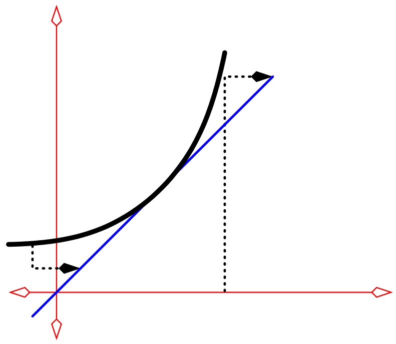
(c) $F(x) = x + x^2$

- A careful plot shows that the neutral fixed point at $x = 0$ is in weakly repelling to the right and weakly attracting to the left:



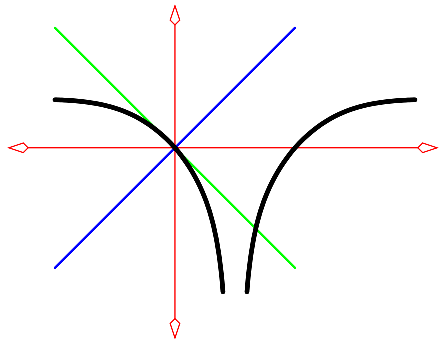
(d) $F(x) = e^{x-1}$

- A careful plot shows that the neutral fixed point at $x = 1$ is in weakly repelling to the right and weakly attracting to the left:



(e) $F(x) = \log|x-1|$

- This one is actually quite hard. The function has a neutral fixed point at $x = 0$. The derivative at that point is -1, and so the function behaves like $x \mapsto -x$ near 0. This means that there is a strong alternating nature to the orbits — they bounce to either side of $x = 0$.



Because of this alternating nature, it is a good idea to plot $F^2(x)$ — this does not move the fixed point, but changes the gradient to $+1$ which makes the analysis easier. The easiest way to get an idea of the shape of the $F(F(x))$ close to $x = 0$ is to look at the derivatives. Now close to $x = 0$, $F(x) \equiv$

$\log(1-x)$, so:

$$\begin{aligned} F(F(x)) &= \log(1-\log(1-x)) \\ \frac{\mathrm{d}F^2}{\mathrm{d}x} &= ((1-x)(1-\log(1-x)))^{-1} \\ \frac{\mathrm{d}}{\mathrm{d}x}\frac{\mathrm{d}F^2}{\mathrm{d}x} &= -\frac{\log(1-x)}{(1-x)^2(1-\log(1-x))^2} \\ \frac{\mathrm{d}}{\mathrm{d}x}\frac{\mathrm{d}}{\mathrm{d}x}\frac{\mathrm{d}F^2}{\mathrm{d}x} &= \frac{1-\log(1-x)-2\log(1-x)^2}{(1-x)^3(1-\log(1-x))^3} \end{aligned}$$

So at $x = 0$, $(F^2)' = 1$, $(F^2)'' = 0$ and $(F^2)''' = 1$. Hence close to $x = 0$ the function is locally $F^2(x) \approx x + x^3/6$. Analysis of this shows that $x = 0$ is then weakly repelling.

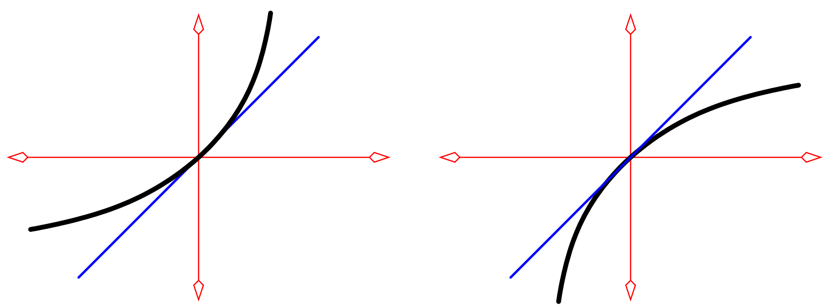
5. Suppose $F(x)$ has a neutral fixed point at x_0, with $F'(x_0) = 1$.

   (a) Suppose that $F''(x_0) > 0$. Is x_0 weakly attracting, repelling or neither?

   (b) Suppose that $F''(x_0) < 0$. Is x_0 weakly attracting, repelling or neither?

   Use graphical analysis and the concavity of F near x_0 to support your answer.

   - Consider the two graphs below:



     The left-hand graph shows a function with $F(0) = 0$, $F'(0) = 1$ and $F''(0) > 0$. Graphical analysis shows that the fixed point is attracting on the left and repelling on the right. Similarly the right-hand graph shows the case $F''(0) < 0$. This graph is attracting on the right and repelling on the left.

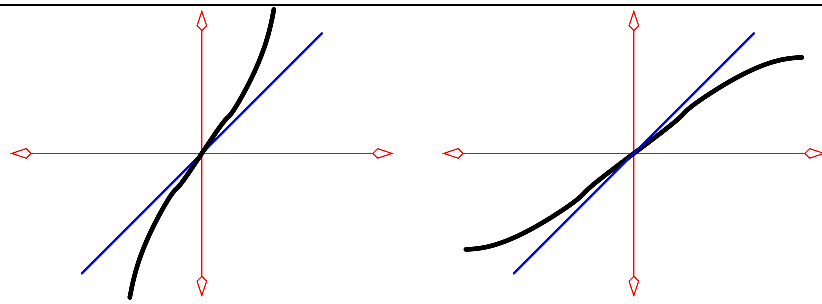
6. Suppose $F(x)$ has a neutral fixed point at x_0. Further $F'(x_0) = 1$ and $F''(x_0) = 0$.

   (a) Show that if $F'''(x_0) > 0$ then x_0 is weakly repelling.

   (b) Show that if $F'''(x_0) < 0$ then x_0 is weakly attracting.

   Again use graphical analysis and the concavity of F to support your answer .

   - Consider the two graphs below:



The left-hand graph shows a function with $F(0) = 0$, $F'(0) = 1$, $F''(0) = 0$ and $F'''(0) > 0$. Graphical analysis shows that the fixed point is weakly repelling. Similarly the right-hand graph shows the case $F'''(0) < 0$. This graph is weakly attracting.