5 Problem Set 5 — Topological conjugacy

1. Find a topological conjugacy between the logistic map

$$F_\mu(x) = \mu x(1-x)$$

and the quadratic map

$$Q_c(x) = x^2 + c$$

Hint: try a linear conjugacy ($h(x) = ax + b$) and equate coefficients of x. You will also have to work out how c and μ are related.

- Let $h(x)$ be a linear function, $h(x) = ax + b$. We now try to solve

$$f(h(x)) = h(Q(x))$$

This gives

$$-\mu a^2 x^2 + \mu a(1-2b)x + \mu b(1-b) = ax^2 + ac + b$$

Equating coefficients of x gives a system of equations involving a, b, c and μ. An obvious choice of b is $b = 1/2$ since this sets the coefficient of x^1 to be zero. We then can choose $a = -1/\mu$ which equates the coefficients of x^2. This then leaves the coefficients of x^0:

$$\mu b(1-b) = ac + b \longrightarrow \mu/4 = 1/2 - c/\mu$$

This shows that $c = -\mu(2-\mu)/4$. Hence the conjugacy between F and Q is given by

$$\begin{aligned} h(x) &= \frac{1}{2} - \frac{x}{\mu} \\ c &= -\frac{\mu(2-\mu)}{4} \end{aligned}$$

Since h is a linear function it is 1-1 so this is a topological conjugacy rather than a semi-conjugacy.

2. Find a conjugacy between the tent-map, $T : [0,1] \mapsto [0,1]$:

$$T(x) = \begin{cases} 2x & x \leq 1/2 \\ 2 - 2x & x > 1/2 \end{cases}$$

and $G(x) = 2x^2 - 1$ on the interval $[-1, 1]$. *Hint*: think “angle doubling”.

- Thanks to the "angle-doubling" hint, we might try something like $h(\theta) = \cos(\theta)$, since $\cos(2\theta) = 2\cos^2\theta - 1$. In particular try $h(x) = \cos(\pi x)$:

$$\begin{aligned} h(T(x)) &= G(h(x)) \\ \cos(2\pi x) &= 2\cos^2(\pi x) - 1 \\ &= \cos(2\pi x) \end{aligned}$$

  as required. However we also have to test the other branch of T:

$$\begin{aligned} h(T(x)) &= G(h(x)) \\ \cos(2\pi - 2\pi x) &= 2\cos^2(2\pi x) - 1 \\ &= \cos(2\pi x) \end{aligned}$$

  which is true since $\cos(2\pi - \theta) = \cos(\theta)$. Now since $h : [0, 1] \mapsto [-1, 1]$ is a 1-1 function, this is a topological conjugacy rather than a semi-conjugacy.

3. Find a conjugacy between $G(x) = 2x^2 - 1$ on $[-1, 1]$ and $Q_{-2}(x) = x^2 - 2$ on $[-2, 2]$. *Hint*: try a linear conjugacy.

   - Try $h : [-1, 1] \mapsto [-2, 2]$ defined by $h(x) = 2x$.

4. Find a conjugacy between the "tripling map" on S^1, $F(\theta) = 3\theta$, and $G(x) = 4x^3 - 3x$ on $[-1, 1]$.

   - Try $h : S^1 \mapsto [-1, 1]$ defined by $h(\theta) = \cos(\theta)$.