6 Problem Set 6 — Symbolic Dynamics

1. Find all points in Σ that are distance exactly 1/2 from the point $(0000\ldots)$.

   - Write $s = (s_0 s_1 s_2 \ldots)$. The distance between s and $(000\ldots)$ is:

     $$d[s,(000\ldots)] = \sum_{k\geq 0} \frac{|s_k|}{2^k} = \sum_{k\geq 0} \frac{s_k}{2^k}$$

     since $s_k = 0,1$. The term s_0 must be 0, otherwise the distance would be bigger than 1. We have 2 choices for s_1.

     – If $s_1 = 1$, then $d[s,(000\ldots)] = 1/2 + \sum_{k\geq 1} \frac{s_k}{2^k}$ and so $s_k = 0$ for all $k \geq 2$.

     – If $s_1 = 0$ then $d[s,(000\ldots)] = \sum_{k\geq 2} \frac{s_k}{2^k}$ which is bounded above by 1/2. Hence we require $s_k = 1$ for all $k \geq 2$.

     Thus we have $s = (01000\ldots)$ or $s = (00111\ldots)$. Use the proximity theorem to see that there are no other points.

2. Find two points halfway between $(000\ldots)$ and $(111\ldots)$. Are there any other such points? Why or why not?

   - The distance between $(00\ldots)$ and $(11\ldots)$ is 2. There are 2 points distance 1 from these points: $s = (100\ldots)$ and $s = (0111\ldots)$. There are no other such points — the proximity theorem will show this.

3. Decide whether or not the following sets are dense in $[0,1]$.

   (a) The set of all numbers in $[0,1]$ except those of the form $1/2^n$, $n = 1,2,3,\ldots$.

   - This set is dense in $[0,1]$. Between any two points in $[0,1]$ there will always be another point in $[0,1]$ that is not of the form $1/2^n$.

   (b) The Cantor middle thirds set.

   - This set is not dense in $[0,1]$ — around $x = 1/2$ there are no points in the Cantor set.

   (c) The compliment of the Cantor middle thirds set.

   - Look at the ternary expansion of a point x in the compliment of the Cantor set — it *must* contain a 1. In any ε-neighbourhood of a point x in $[0,1]$ we can find a point whose ternary expansion has a 1 in it — just put a 1 far enough down the expansion and it will differ from x by a sufficiently small amount. Hence the compliment of the cantor set is dense in $[0,1]$.

4. Is the orbit of the point $(01\ 001\ 0001\ 00001\ \ldots)$ under σ dense in Σ?

- No — there is no point in this orbit close to $(11111\ldots)$.

The following questions concern the space of sequences on N symbols, Σ_N, together with the shift map σ_N and the distance function:

$$d[s,t] = \sum_{k=0}^{\infty} \frac{|s_k - t_k|}{N^k}$$

5. Prove that $\sigma_N : \Sigma_N \mapsto \Sigma_N$ is continuous.

   - The proof is analogous to the proof that σ is continuous on sequences of 2 symbols. Take a point $x \in \Sigma_N$ and an $\varepsilon > 0$. Pick n such that $1/N^n < \varepsilon$. The proximity theorem (on N symbols) then says that all points whose first $n+1$ terms agree with x will be within ε of x. Pick $\delta = 1/N^{n+1}$ — points that are withing δ of x will have their first $n+2$ terms the same. Applying the shift map to these points will give points that agree with the first $n+1$ terms of $\sigma(x)$. Hence we have — given x and ε there exists a δ such that if $d[x,t] < \delta$ then $d[\sigma_N(x), \sigma_N(\delta)] < \varepsilon$, and so σ_N is continuous at x. Since x was arbitary we have that σ_N is continuous on all of Σ_N.

6. How many points of *prime*-period 2 does σ_N have?

   - There are N fixed points under σ_N — these are

     $$(000\ldots) \qquad (111\ldots) \qquad \ldots \qquad ((N-1)(N-1)(N-1)\ldots)$$

     There are N^2 fixed points of σ^2 of which N will be fixed points of σ. Hence there are $N^2 - N$ points of prime period 2.

7. Define the new distance function:

   $$d_\delta[s,t] = \sum_{k=0}^{\infty} \frac{\delta_k(s,t)}{N^k}$$

   where

   $$\delta_k(s,t) = \begin{cases} 1 & \text{if } s_k \neq t_k \\ 0 & \text{if } s_k = t_k \end{cases}$$

   Prove that $d_\delta[s,t]$ is also a metric on Σ_N.

   - We need to show a few things:
     - Since $\delta_k(s,t) = \delta(t,s)$ we have that $d_\delta[s,t] = d_\delta[t,s]$.
     - Since the sum that defines d_δ consists of terms that are non-negative it follows that $d_\delta[s,t] \geq 0$.

  - If $s_k = t_k$ for all k — then $d_\delta[s,t] = \sum_{k\geq 0} 0 = 0$. While if $d_\delta[s,t] = 0$ it follows (because each term is ≥ 0) that each term in the sum must be 0 and so $s_k = t_k$ for all k. We have shown that $d_\delta[s,t] = 0 \leftrightarrow s = t$.
  - Last of all we need to show the triangle inequality:

$$\begin{aligned} d_\delta[s,t] &= \sum_{k=0}^{\infty} \frac{\delta_k(s,t)}{N^k} \\ &\leq \sum_{k=0}^{\infty} \frac{\delta_k(s,u) + \delta(u,t)}{N^k} \\ &= d_\delta[s,u] + d_\delta[u,t] \end{aligned}$$

    The middle line follows because:
    * if s_k, t_k are the same, then $\delta_k(s,t) = 0$.
    * if s_k, t_k are different then $\delta_k(s,t) = 1$ and one or both of the $\delta_k(s,u)$ and $\delta_k(u,t)$ must also be 1.

8. Using $d_\delta[s,t]$ what is the maximum distance between two points in Σ_N?

   - The maximum distance will be obtained by maximising each term in the sum that defines d_δ:

$$d_\delta[s,t] = \sum_{k=0}^{\infty} \frac{\delta_k(s,t)}{N^k} \leq \sum_{k=0}^{\infty} \frac{1}{N^k}$$

     This last sum is $N/(N-1)$. One can verify that this distance is obtained by taking the sequences $(000\ldots)$ and $(111\ldots)$.