6 Symbolic Dynamics — summary

6.1 Sequence space

Definition. The sequence space on two symbols is the set

$$\Sigma = \{(s_0 s_1 s_2 \ldots) \mid s_k = 0 \text{ or } 1\}.$$

The space Σ is the set of all infinite sequences of "0" s and "1" s. You should not think of the elements of Σ as numbers — they are not, rather they are infinite "words" in two letters which we write as 0 and 1. We could equally well use "a" and "b".

Definition. Let $s = (s_0 s_1 s_2 \ldots)$ and $t = (t_0 t_1 t_2 \ldots)$ be two elements in Σ. We define the distance between them to be

$$d[s,t] = \sum_{k \geq 0} \frac{|s_k - t_k|}{2^k}.$$

This function, $d[s,t]$ is a generalisation of the usual concept of distance. Such functions are called metrics.

Definition. A function d is called a *metric* on a set X if for all $x, y, z \in X$ the following hold:

1. $d[x,y] \geq 0$ and further $d[x,y] = 0$ if and only if $x = y$.
2. $d[x,y] = d[y,x]$.
3. $d[x,z] \leq d[x,y] + d[y,z]$.

Given that we now have a concept of distance on Σ given by $d[s,t]$, we want to get an idea of what it means for two points in Σ to be close together

Theorem (The proximity theorem). Let $s, t \in \Sigma$.

1. If $s_k = t_k$ for $k = 0, 1, \ldots, n$ then $d[s,t] \leq 1/2^n$.
2. If $d[s,t] < 1/2^n$ then $s_k = t_k$ for $k = 0, 1, \ldots, n$.

This theorem says that two points are close together if their first few terms agree.

6.2 The shift map

We are unable to do much with sequences in Σ. We have no idea how to add them or multiply them, etc. etc. Perhaps the simplest thing we can do to a sequence in Σ is to cut off the first term. This defines the shift map:

Definition. The shift map $\sigma : \Sigma \mapsto \Sigma$ is:

$$\sigma(s_0 s_1 s_2 \ldots) = (s_1 s_2 s_3 \ldots)$$

The map is easily iterated:

$$\sigma^k(s_0 s_1 s_2 \ldots) = (s_k s_{k+1} s_{k+2} \ldots)$$

We can now define a dynamical system on Σ using the shift map. We can look for periodic points etc, just as we have for other dynamical systems.

The fixed points under σ are simply the sequences of all 0 and all 1:

$$(000\ldots) \text{ and } (111\ldots)$$

The periodic points of period n must be of the form:

$$(s_0 s_1 s_2 \ldots s_{n-1}\ s_0 s_1 \ldots s_{n-1} \ldots) = (\overline{s_0 s_1 \ldots s_{n-1}})$$

Hence there are 2^n points of period n — note that some of these are certainly going to have prime period less than n.

Similarly one can find eventually fixed and eventually periodic points.

Theorem. The shift map is a continuous function.

A reminder about continuity:

Definition. Let $F : X \mapsto X$ be a function on a set X with a metric d. F is continuous at $x_0 \in X$ if $\forall \epsilon > 0$ there is a $\delta > 0$ such that if $d[x, x_0] < \delta$ then $d[F(x), F(x_0)] < \epsilon$. If F is continuous at all points in X then we say that F is a continuous function.

The proof of the theorem is not too complicated. If two points are distance ϵ apart in Σ then their first $n+1$ terms must be the same for $1/2^n < \epsilon$. If we go backwards then *before* we apply the shift map to these two points their first $n+2$ terms need to agree in order for them to be close together…and so that when we apply the shift map the first $n+1$ terms still agree.

Hence if we pick $\delta < 1/2^{n+1}$ then any two points that are distance δ apart must agree in their first $n+2$ terms. Applying σ to these points mean that their first $n+1$ terms must agree (since we have cut off the first element of each sequence). This means that they are a distance $< 1/2^n$ apart. We picked n so that $1/2^n < \epsilon$. Hence we are done.

6.3 Three properties of a chaotic system

Definition (Devaney). Devaney gives the following definition of a chaotic dynamical system. A dynamical system $F : X \mapsto X$ is *chaotic* if:

1. Periodic points of F are dense in X.
2. F is transitive.
3. F depends sensitively on initial conditions.

Definition. Let X be a set and let Y be a subset of X. Y is a dense subset of X if for any point $x \in X$ and any $\epsilon > 0$ there is a point $y \in Y$ a distance less than ϵ from x.

Lemma. The periodic points of σ form a dense subset of Σ.

Proof. Pick a point $x = (x_0 x_1 x_2 \ldots)$ and an $\epsilon > 0$. Pick n such that $1/2^n < \epsilon$. If we construct a new point $y = (y_0 y_1 y_2 \ldots)$ such that $y_k = x_k$ for $k = 0, 1, \ldots, n$, then by the proximity theorem the distance between x and y is less than $1/2^n$.

So in order for the set of periodic points to be dense in Σ we need to construct a periodic point within ϵ of x — hence we need a periodic point, y, whose first $n+1$ terms agree with those of x. Hence let $y = (\overline{x_0 x_1 x_2 \ldots x_n})$ (*i.e.*repeat the first $n+1$ terms of x). It is clear that y is periodic and by the proximity theorem it is within ϵ of x.

Definition. A dynamical system is transitive if for any pair of points $x, y \in X$ and any $\epsilon > 0$ there is another point $z \in X$ such that the orbit of z comes within ϵ of both x and y.

Lemma. The shift map is transitive.

Proof. Pick any x and y in Σ and an $\epsilon > 0$. We will construct a z whose orbit comes within ϵ of both x and y. Pick n such that $1/2^n < \epsilon$. This means that if a new point has its first $n+1$ terms the same as those of x then by the proximity theorem it is within ϵ of x (similarly for y).

Construct the following sequence:

$$z = (\underbrace{\underline{0\ 1}}_{\text{1 blocks}}\quad \underbrace{\underline{00\ 01\ 10\ 11}}_{\text{2 blocks}}\quad \underbrace{\underline{000\ 001\ldots}}_{\text{3 blocks}}\quad \ldots)$$

That is we glue together all sequences of length 1 and then all sequences of length 2 etc etc. Hence any sequence of length n will appear in z at some point.

Now consider the first $n+1$ terms of x. These are a sequence of length $n+1$ — hence they must (by construction) appear somewhere in z. Thus there is a k_1 such that the first $n+1$ terms of $\sigma^{k_1}(z)$ are the same as those of x — hence the orbit of z comes within ϵ of x. Similarly there exists k_2 such that the first $n+1$ terms of $\sigma^{k_2}(z)$ are the same as those of y. Hence the orbit of z comes within ϵ of both x and y.

Definition. A dynamical system F *depends sensitively on initial conditions* if there is a $\beta > 0$ such that for any $x \in X$ and $\epsilon > 0$ there exists a $y \in X$ within ϵ of x and a $k > 0$ such that $F^k(x)$ and $F^k(y)$ are at least β apart.

This definition says that there is some "error bar" $= \beta$. Given this error bar no matter which x we pick and which little region around it, we can find a new point y in this region so that the orbits of x and y will begin to diverge, and at some point they will be at least β apart.

Lemma. The dynamical system defined by the shift map depends sensitively on initial conditions.

Proof. Put $\beta = 1$. Now pick any $x \in \Sigma$ and any $\epsilon > 0$. Then pick n such that $1/2^n < \epsilon$. We now pick a point close (within ϵ) of x and show that its orbit will diverge from that of x by at least β.

Pick $y \neq x$, such that $d[x, y] < 1/2^n$. This means that the first $n + 1$ terms of x and y are the same. Because $x \neq y$ there exists $k > n$ such that $x_k \neq y_k$. Consider now the points $\sigma^k(x)$ and $\sigma^k(y)$. The first term $\sigma^k(x)$ is x_k and the first term of $\sigma^k(y)$ is $y_k \neq x_k$. Hence the distance between these two points is:

$$d[\sigma^k(x), \sigma^k(y)] = \sum_{n \geq 0} \frac{|s_{n+k} - t_{n+k}|}{2^n} \geq \frac{|s_k - t_k|}{2^0} = 1$$

Thus the orbits of x and y diverge by at least β.

Combining the previous three lemmas gives:

Theorem. The shift map defines a chaotic dynamical system.