5 Topological conjugacy — summary

Definition. Some basic function definitions:

- A function $f : X \mapsto Y$ is *1 to 1* or *injective* if:
  - if $f(a) = f(b)$ then $a = b$.
  - equivalently, if $a \neq b$ then $f(a) \neq f(b)$.
  - That is f maps distinct elements to distinct elements.
- A function $f : X \mapsto Y$ is *onto* or *surjective* if:
  - For all $y \in Y$ there exists $x \in X$ such that $f(x) = y$.
  - That is, every element in Y has at least one element in X that maps to it under f. Every element in Y has a preimage in X.

Definition. A function $h : X \mapsto Y$ is a *homeomorphism* if

- h is 1 to 1 (or injective)
- h is onto (or surjective)
- h is continuous
- h^{-1} is also continuous.

Definition. Let $F : X \mapsto X$ and $G : Y \mapsto Y$ be two functions. We say that F and G are *topologically conjugate* if there exists a homeomorphism $h : X \mapsto Y$ such that:

$$h \circ F = G \circ h \qquad \text{or equivalently,} \qquad h(F(x)) = G(h(y))$$

Such a homeomorphism h is called a *topological conjugacy.*

- Let us define two different dynamical systems using $F : X \mapsto X$ and $G : Y \mapsto Y$.
- If F and G are topologically conjugate, then the dynamics they define on X and Y are equivalent, and the conjugacy h shows how orbits in one set map to orbits in the the other set.
- Consider the orbit of a point x_0 under F:

$$\{x_0, x_1 = F(x_0), x_2 = F^2(x_0), x_3 = F^3(x_0), \dots\}$$

- Let $h : X \mapsto Y$ be a conjugacy that satisfies:

$$h \circ F = G \circ h \qquad \text{or} \qquad h(F(x)) = G(h(x))$$

- Then we can find the image of x_0 under h

$$y_0 = h(x_0).$$

  and define its orbit in Y under G:

$$\{y_0, y_1 = G(y_0), y_2 = G^2(y_0), y_3 = G^3(y_0), \dots\}$$

- These two orbits are equivalent under h. That is $y_k = h(x_k)$ for all $k \geq 0$.

- Consider the point y_n. We can obtain it in two different ways:

$$y_n = h(x_n) = h(F^n(x_0))$$

  and also

$$y_n = G^n(y_0) = G^n(h(x_0))$$

  or a bit more pictorially:

$$\begin{array}{ccccccc} x_0 \in X & \xrightarrow{F} & x_1 \in X & \xrightarrow{F} & \dots & \xrightarrow{F} & x_n \in X \\ \downarrow h & & \downarrow h & & & & \downarrow h \\ y_0 \in Y & \xrightarrow{G} & y_1 \in Y & \xrightarrow{G} & \dots & \xrightarrow{G} & y_n \in Y \end{array}$$

- So we need to show that $h(F^n(x_0)) = G^n(h(x_0))$. By the conjugacy condition given above, we can rewrite this as:

$$\begin{aligned} G^n(h(x_0)) &= G^{n-1}(G(h(x_0))) \\ &= G^{n-1}(\ h(F(x_0))) \\ &= G^{n-2}(\ G(\ h(F(x_0)))) = G^{n-2}(\ h(F^2(x_0))) \\ &\quad \text{keep “pushing } h \text{ to the left} \\ &= G^{n-k}(\ h(F^k(x_0))) \\ \dots & \\ &= h(F^n(x_0)) \end{aligned}$$

  as required.

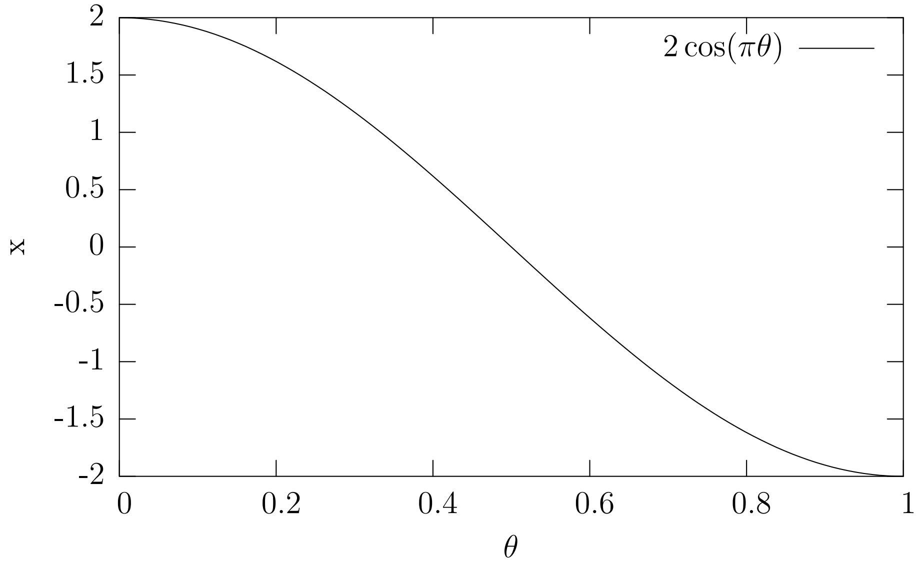
- We can similarly get back from Y and G to X and F using the inverse of h.

Example 1. There is a topological conjugacy between $Q_{-2}(x) = x^2 - 2$ on $[-2, 2]$ and the tent map on $[0, 1]$:

$$T(\theta) = \begin{cases} 2\theta & 0 \le \theta < 1/2 \\ 2 - 2\theta & 1/2 \le x \le 1 \end{cases}$$

Proof:

- Define $h : [0, 1] \mapsto [-2, 2]$ by
$$h(\theta) = 2\cos\pi\theta$$
- We need to show that $Q(h(\theta)) = h(T(\theta))$.
- Let us start with the LHS:
$$\begin{aligned} Q(h(\theta)) &= 4\cos^2\pi\theta - 2 \\ &= 2\cos 2\pi\theta \\ &= h(2\theta) \end{aligned}$$
- If we now consider the RHS — since T is made up of two 2 different linear functions we need to check both.
- If $\theta < 1/2$ then $T(\theta) = 2\theta$ and
$$h(T(\theta)) = h(2\theta) = Q(h(\theta))$$
as required.
- If $\theta \ge 1/2$ $T(\theta) = 2 - 2\theta$ and we need to do a little more work:
$$\begin{aligned} h(T(\theta)) &= h(2 - 2\theta) \\ &= 2\cos(2\pi - 2\pi\theta) \\ &= 2\cos(-2\pi\theta) \\ &= 2\cos(2\pi\theta) = h(2\theta) = Q(h(\theta)) \end{aligned}$$
as required.
- We still need to make sure that h is a homeomorphism.
  - h is continuous and so is h^{-1}.
  - h is onto — every element of $[-2, 2]$ is mapped to by an element of $[0, 1]$.
  - h is 1 to 1 — distinct elements of $[0, 1]$ map to distinct elements of $[-2, 2]$ — check the following plot


2 cos(πθ)
2
1.5
1
0.5
0
-0.5
-1
-1.5
-2
x
0
0.2
0.4
0.6
0.8
1
θ