Random subgroups of Thompson's group F

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Abstract

We consider random subgroups of Thompson's group F with respect to two natural stratifications of the set of all k generator subgroups of this group. We find that the isomorphism classes of subgroups which occur with positive density vary greatly between the two stratifications. We give the first known examples of *persistent* subgroups, whose isomorphism classes occur with positive density within the set of k-generator subgroups, for all k greater than some k_0. Additionally, Thompson's group provides the first example of a group without a generic isomorphism class of subgroup. In F, there are many isomorphism classes of subgroups with positive density less than one. Elements of F are represented uniquely by reduced pairs of

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finite rooted binary trees. We compute the asymptotic growth rate and a generating function for the number of reduced pairs of trees, which we show is D-finite and not algebraic.

1 Introduction

A main focus in the field of asymptotic group theory is to quantify precisely the meaning of the statement that "most" groups of a particular form have (or do not have) a certain property. In this paper we investigate a related question– the likelihood of randomly selecting a particular subgroup of a given group G, up to isomorphism, from the set of all k-generator subgroups of G. We take the group G to be Thompson's group F. To define the density of an isomorphism class of subgroup within the set of all k-generator subgroups of F, we introduce two natural stratifications of this set into spheres of size n. Intuitively, the density of an isomorphism class of subgroups is the probability that a randomly selected k-generator subgroup is in the class. We find that the isomorphism classes of subgroups which occur with positive density vary greatly between the two stratifications. We give the first known examples of *persistent* subgroups, whose isomorphism classes occur with positive density within the set of all k-generator subgroups, for all k greater than some k_0. Additionally, Thompson's group provides the first example of a group without a generic type of subgroup. That is, there are infinitely many isomorphism classes of subgroups with positive density less than one.

The likelihood of a particular subgroup of a given group to be selected at random is motivated by questions in group-based cryptography, and has applications in this field. The analysis of the security of algorithms used in cryptography can depend upon the expected isomorphism type of a random subgroup. Our definition of the asymptotic density of a particular subgroup H of a group G follows Borovik, Miasnikov and Shpilrain in [1]. In this paper, they present a detailed discussion of asymptotic and statistical questions in group theory. We also refer the reader to Kapovich, Miasnikov, Schupp and Shpilrain [13] for background on generic-case complexity and notions of density.

Let G be an infinite group and X a set of representatives of elements that maps onto G. We can associate to each $x \in X$ an integer *size*. For example, let X be the set of all words in a finite generating set for G, with size corresponding to word length. Let X_k be the set of unordered k-tuples of representatives $x \in X$. Then each member of X_k corresponds to a k-generated subgroup of G, taking the k representatives as the generators. Assume that a notion of size has been fixed on X. We can define an integer size for each k-tuple in a variety of ways. For example, the size of a k-tuple could be the sum of the sizes of its components. Alternately, one could take the size of a k-tuple to be the maximum size of any of its components. Once a notion of size is fixed, the set of all tuples of size n in X_k is called the n-sphere, and denoted $\mathrm{Sph}_k(n)$. Such a decomposition of X_k into spheres of increasing radii is known as a *stratification* of X_k. We prefer our spheres of a fixed size to be finite and thus can regard these spheres of increasing radii as an exhaustion of an infinite set X_k by a collection of finite sets.

To quantify the likelihood of randomly selecting a particular subset of X_k, we take a limit of the counting measure on spheres of increasing radii. The *asymptotic density* of a subset T in X_k is defined to be the limit

$$\lim_{n\to\infty} \frac{|T \cap \mathrm{Sph}_k(n)|}{|\mathrm{Sph}_k(n)|}$$

if this limit exists, where $|T|$ denotes the size of the set T. We often omit the word asymptotic and refer to this limit simply as the density of T.

Let T_H be the set of k-tuples that generate a subgroup of G isomorphic to some particular subgroup H. If the density of T_H is positive we say that H is *visible* in the space of k-generated subgroups of G. We call the set of all visible k-generated subgroups of G the *k-subgroup spectrum*, denoted by $\mathrm{Spec}_k(G)$. If the density of T_H is one, we say that H is *generic*; if this density is zero we say that H is *negligible*.

We have made a series of choices within this construction, each of which can greatly influence the densities of different subsets; those choices include: the representation of group elements, the size function defined on X, and the stratification of the set X_k. Additionally, we are asserting that the likelihood of randomly selecting a k-generator subgroup isomorphic to the given one is captured by the limit as defined. It is certainly possible to construct contrived stratifications which have desired pathological properties. For example, we can construct a stratification of the integers where the prime numbers have asymptotic density one by choosing the sphere of size n to be the $(2^{n-1}+1)$th through 2^nth prime numbers and the single nth composite number. Thus we concentrate on stratifications which correspond to a natural definition of the sphere of size n in X_k. We will show below that for Thompson's group F, a small change in the stratification has great impact on the set of visible subgroups.

In this article we consider asymptotic densities of isomorphism classes of subgroups of Thompson's group F. This group presents the first example of a group which has different asymptotic properties with respect to two different, yet natural, methods of stratification. To define these stratifications, we represent elements of F using reduced pairs of finite rooted binary trees, which we abbreviate to "reduced tree pairs". These representatives are in one-to-one correspondence with group elements. Each pair consists of two finite, rooted binary trees with the same number of leaves, or equivalently, with the same number of internal nodes or carets, as defined below. The size of a tree pair will be the number of carets in either tree of the pair.

Using these reduced tree pairs to represent elements of F, we define the sphere of radius n in X_k in two natural ways:

1. take $\mathrm{Sph}_k(n)$ to be the set of k-tuples in which the sum of the sizes of the coordinates is n, or

2. take $\mathrm{Sph}_k(n)$ to be the set of k-tuples where the maximum size of a coordinate

is n.

With respect to the sum stratification, every non-trivial isomorphism class of m-generated subgroup for $m \leq k$ is visible, that is, has non-zero density. With respect to the "max" stratification, this is not the case.

Another natural stratification to consider on F, or on any finitely generated group, is obtained by taking the size of an element of F to be the word length with respect to a particular set of generators. For F we can consider word length with respect to the standard finite generating set $\{x_0, x_1\}$. This stratifies the group itself into metric spheres. Despite work of José Burillo [3] and Victor Guba [9] in this direction, the sizes of these spheres have not been calculated, and thus it is not computationally feasible to consider the induced stratification of X_n.

It is striking in our results below that the k-generated subgroups of Thompson's group F have no generic isomorphism type with respect to either stratification. All other groups which have been studied in this way exhibit a generic type of subgroup with respect to their natural stratifications. Jitsukawa [12] proved that k elements of any finite rank free group generically form a free basis for a free group of rank k. This property was shown by Miasnikov and Ushakov [14] to be shared by pure braid groups and elementary torsion free hyperbolic groups. To obtain our results on random subgroups of Thompson's group F, we consider group elements as pairs of reduced finite rooted binary trees. We must be able to count the number of pairs of trees of this form of a given size. Woodruff [18] studied this in his thesis, and conjectured that the number of reduced tree pairs of size n is proportional to $(8+4\sqrt{3})^n/n^3$. We prove Woodruff's conjectured growth rate, and additionally show that the generating function for the number of reduced tree pairs of size n is D-finite, meaning that it satisfies a particular type of differential equation, but not algebraic.

This paper is organized as follows. In Section 2, we consider the number of pairs of reduced trees of size n, which we call r_n. We prove that r_n has a D-finite generating function which is not algebraic. We prove that r_n approaches $A\mu^n/n^3$ uniformly, where A is a constant and $\mu = 8 + 4\sqrt{3} \approx 14.93$.

In Section 3 we describe Thompson's group F as well as particular subgroups of this group which will be important in later sections.

In Section 4 we compute the asymptotic density of isomorphism classes of k-generator subgroups with respect to the *sum stratification*. We prove that if G is a non-trivial m-generator subgroup, then its isomorphism class is visible in the space of k-generator subgroups for $k \geq m$. In contrast to previously known examples, no subgroup type is generic in this stratification.

In Section 5 we compute the asymptotic density of isomorphism classes of k-generator subgroups with respect to the *max stratification* and find very different behavior. In this case, not every subgroup is visible for large k. For example, we prove that $\mathbb{Z}$ is visible in the set of k-generated subgroups only for $k = 1$. Yet there are examples of isomorphism classes of subgroups which are *persistent*; that is, visible in the set of k generator subgroups for all sufficiently large k. For example, we show that the isomorphism class of F itself is visible in the set of k-generated subgroups for all $k \geq 2$.

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2 Combinatorics of reduced tree pairs

A *caret* is a pair of edges that join two vertices to a common parent vertex and looks like $\wedge$. An n-caret *tree pair diagram*, or *tree pair* for short, is an ordered pair consisting of two rooted binary trees, each having n carets. A 5-caret tree pair is shown in Figure 1(a). A *leaf* is a vertex of degree one. A tree with n carets will

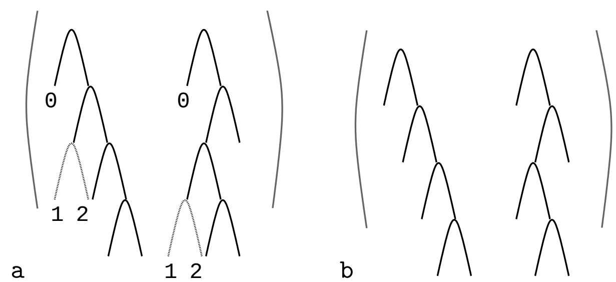

Figure 1: 5-caret unreduced tree pair, and 4-caret reduced tree pair.

have $n+1$ leaves. In the trees we consider, all vertices other than the leaves and the root have degree three. The *left child* of a caret is the caret attached to its left leaf; the *right child* is defined analogously. An *exposed caret* is a caret both of whose children are leaves. A pair of trees with at least two carets in each tree is *unreduced* if, when the leaves are numbered from left to right, each tree contains a caret with leaves numbered k and $k+1$. In an unreduced tree, the caret with identical leaf numbers is removed from both trees, the leaves are renumbered, and the trees are again inspected for possible reductions. For example, the tree pair in Figure 1(a) is unreduced. Removing the exposed caret with leaves labeled $1, 2$ in each tree gives the reduced tree pair in Figure 1(b). A pair of trees which is not unreduced is called reduced. Note that we do not reduce a pair of single carets: we insist that our tree pairs are always nonempty. We denote the number of reduced n-caret tree pairs by r_n, so we have $r_0 = 0$ and $r_1 = 1$.

Ben Woodruff studied the enumeration of $\{r_n\}$ in his thesis [18] where he derived a formula for r_n (which he denoted N_n), proved an upper bound of $(8 + 4\sqrt{3})^n \approx 14.93^n$ and conjectured an asymptotic growth rate of $(8+4\sqrt{3})^n/n^3$. We take a different approach to counting r_n and derive a recursive formula in terms of c_n^2, where $c_n = \frac{1}{n+1}\binom{2n}{n}$ is the n-th Catalan number. Working in terms of generating functions for r_n and c_n^2, we obtain a finite order differential equation which leads to a finite polynomial recurrence for r_n. From this we are able to prove the growth rate conjectured by Woodruff. The key to this section is to show that the generating function for r_n is closely related to that for c_n^2 and many of the properties of the generating function for c_n^2 are inherited by that of r_n.

We let $f(k,m)$ denote the number of ordered k-tuples of possibly empty rooted binary trees using a total of m carets, which we call *forests*. So for example $f(3,2)$, which is the number of forests of three trees containing a total of two carets, is equal to nine, as shown by Figure 2. One can prove that $f(k,n) = \frac{k}{2n+k}\binom{2n+k}{n}$

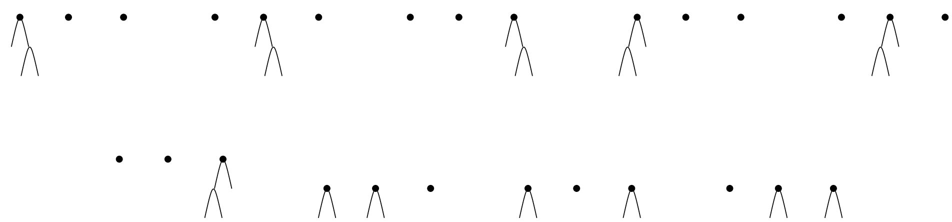

Figure 2: $f(3,2) = 9$.

although we will not need to make use of this formula in this article.

The n-th Catalan number c_n counts the number of binary trees consisting of n carets, and thus c_n^2 is the number of ordered pairs of rooted binary trees with n carets in each tree. Some of these pairs will be reduced, and some not. For those that are not reduced, we can cancel corresponding pairs of carets to obtain an underlying reduced tree pair. In a reduced tree pair of i carets, each tree has $i+1$ leaves. We describe a process which is the inverse of reduction, called "decoration." To decorate a reduced tree pair diagram (S,T) with i carets in each tree, we take a forest of $i+1$ trees and $n-i$ carets, duplicate it, then append the trees in the forests to the corresponding leaves of S and T. The first tree in the forest is appended to the first leaf, the second tree in the forest to the second leaf and so on. We can do this in $f(i+1,n-i)$ different ways. This decorating process yields a new unreduced tree pair with n carets, that will reduce to the original reduced tree pair $(S<T)$ with i carets. For example, the reduced 2-caret tree pair in Figure 3 can be decorated in 9 different ways to give unreduced pairs of 5 carets which would all reduce to the original tree pair diagram. This leads to the

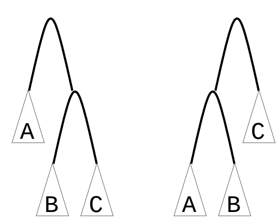

Figure 3: Decorating a reduced tree with a forest of three trees A, B and C.

following lemma.

Lemma 1 (Relating r_n and c_n^2) *For $n \geq 1$*

$$c_n^2 = r_n + r_{n-1}f(n,1) + r_{n-2}f(n-1,2) + \ldots + r_1 f(2, n-1).$$

Proof: Each n-caret tree pair is either reduced or must reduce to a unique reduced tree pair of i carets for some $i \in [1, n-1]$. Hence the total number of n-caret tree pairs, c_n^2, is the number of pure reduced pairs of n-carets, r_n, plus the number of reduced i-caret tree pairs, r_i, multiplied by the number of ways to decorate them with a forest of $n-i$ carets, $f(i+1, n-i)$, for each possible value of i. □

We can reformulate this recursion in terms of generating functions. We define the generating functions for r_n, c_n and c_n^2 as:

$$\begin{array}{rcl} R(z) = & & r_1 z + r_2 z^2 + \ldots \\ C(z) = & c_0 + & c_1 z + c_2 z^2 + \ldots \\ P(z) = & & c_1^2 z + c_2^2 z^2 + \ldots \end{array}$$

Note that $R(z)$ and $P(z)$ have no constant term while $C(z)$ does. We prove in the following proposition that $R(z)$ can be obtained from $P(z)$ via a simple substitution. Using knowledge of $P(z)$ we can find a closed form expression for $R(z)$ and asymptotic growth rate for r_n. Note that if $G(z)$ is the generating function for a set of objects, then $G(z)^k$ is the generating function for ordered k-tuples of those objects. So we can express the generating function of $f(k,n)$ for fixed k as $C(z)^k$.

Proposition 2 (Relating $R(z)$ and $P(z)$) *The generating functions for r_n and c_n^2 are related by the following equation:*

$$R(z) = (1-z)P(z(1-z)),$$

which is equivalent to

$$P(x) = C(x)R(xC(x)).$$

Proof: The generating function for the Catalan numbers is well known and may be written in closed form as

$$C(x) = \frac{1-\sqrt{1-4x}}{2x};$$

it satisfies the algebraic equation $C(x)(1-xC(x)) = 1$. See for example [16]. If we rewrite the equation $R(z) = (1-z)P(z(1-z))$ substituting the variable z with $xC(x)$ then we obtain

$$R(xC(x)) = (1-xC(x))P\big(xC(x)(1-xC(x))\big) = \frac{1}{C(x)}P(x)$$

which rearranges to

$$P(x) = C(x)R(xC(x))$$

This substitution is invertible by $x \mapsto z(1-z)$, and so proving this statement implies that statement of the proposition. By examining the coefficients of x^n we will show that this statement is equivalent to Lemma 1.

The right hand side can be written as

$$C(x)R(xC(x)) = C(x)\sum_{k=0}^{\infty} r_k \big(xC(x)\big)^k = \sum_{k=0}^{\infty} r_k x^k \big(C(x)\big)^{k+1}$$

We will use the notation $[x^i]G(x)$ to denote the coefficient of x^i in the expansion of a generating function $G(x)$. Considering the above equation in terms of the coefficient of x^n we have

$$\begin{aligned} c_n^2 = [x^n]P(x) &= [x^n]C(x)R(xC(x)) \\ &= [x^n]\sum_{k=0}^{\infty} r_k x^k \big(C(x)\big)^{k+1} = \sum_{k=0}^{\infty}[x^n]x^k r_k \big(C(x)\big)^{k+1} \\ &= \sum_{k=0}^{n}[x^{n-k}]r_k \big(C(x)\big)^{k+1} \end{aligned}$$

As noted above, $\big(C(x)\big)^{k+1}$ is the generating function for the number of ordered $(k+1)$-tuples of rooted binary trees, which are counted by $f(k+1,n)$. Thus the coefficient of x^{n-k} in $\big(C(x)\big)^{k+1}$ is precisely $f(k+1,n-k)$, that is, $[x^{n-k}]\big(C(x)\big)^{k+1} = f(k+1,n-k)$. So the above equation becomes

$$c_n^2 = r_n f(n+1,0) + r_{n-1} f(n,1) + \ldots r_1 f(2,n-1) + r_0 f(1,n)$$

which is precisely Lemma 1 since $f(n+1,0)=1$ and $r_0 = 0$. $\square$

A function is said to be *D-finite* if it satisfies a homogeneous linear ordinary differential equation with polynomial coefficients, for example, see [16]. The class of D-finite functions strictly contains the class of algebraic (and rational) functions. If one has a differential equation for a generating function it is often possible to obtain the asymptotic growth rate of its coefficients by studying this differential equation. Following [16], a generating function is D-finite if and only if its coefficients satisfy a finite polynomial recurrence.

Lemma 3 ($R(z)$ is D-finite) *The generating function $R(z)$ satisfies the follow-*

ing linear ordinary differential equation

$$\begin{aligned} z^2(1-z)(16z^2-16z+1)(2z-1)^2\frac{\mathrm{d}^3R}{\mathrm{d}z^3} \\ - z(2z-1)(16z^2-16z+1)(8z^2-11z+5)\frac{\mathrm{d}^2R}{\mathrm{d}z^2} \\ - (128z^5-320z^4+365z^3-232z^2+76z-4)\frac{\mathrm{d}R}{\mathrm{d}z} \\ + 36z(z-1)R(z) = 0. \end{aligned}$$

It follows that $R(z)$ is D-finite.

Proof: Starting from a recurrence satisfied by the Catalan numbers we can find a differential equation satisfied by $P(z)$ and then standard tools allow us to transform this equation into one satisfied by $R(z)$.

Since $c_n = \frac{1}{n+1}\binom{2n}{n}$, we have the following recurrence for the Catalan numbers:

$$(n+2)c_{n+1} = 2(2n+1)c_n.$$

Squaring both sides yields

$$(n+2)^2c_{n+1}^2 = 4(2n+1)^2c_n^2.$$

Thus we have a finite polynomial recurrence for the coefficients of $P(z)$, which means that we can find a linear differential equation for P(z). We do this using the Maple package `GFUN` [15] to obtain

$$(z^2-16z^3)\frac{\mathrm{d}^2P}{\mathrm{d}z^2} + (3z-32z^2)\frac{\mathrm{d}P}{\mathrm{d}z} + (1-4z)P(z) = 1.$$

The original recurrence can be recovered by extracting the coefficient of z^n in the above equation. We can then make this differential equation homogeneous

$$(16z^3-z^2)\frac{\mathrm{d}^3P}{\mathrm{d}z^3} + (80z^2-5z)\frac{\mathrm{d}^2P}{\mathrm{d}z^2} + (68z-4)\frac{\mathrm{d}P}{\mathrm{d}z} + 4P(z) = 0.$$

Making the substitution $z \mapsto z(1-z)$ using the command `algebraicsubs()` in `GFUN` we find a differential equation satisfied by $P(z(1-z))$. This in turn leads to the homogeneous differential equation for $R(z)$ given above. □

Following the notation of Flajolet [8], we say that two functions are *asymptotically equivalent* and write $f(n) \sim g(n)$ when

$$\lim_{n\to\infty}\frac{f(n)}{g(n)} = 1$$

.

Proposition 4 (Woodruff's conjecture) *r_n is asymptotically equivalent to (has an asymptotic growth rate of) $A\mu^n/n^3$ where $\mu = 8+4\sqrt{3}$ and $A > 0$ is a constant. Explicitly this means that the limit*

$$\lim_{n\to\infty} \frac{r_n}{A\mu^n n^{-3}} = 1.$$

Proof: We start by finding a very rough bound on the exponential growth of r_n. We then refine this by analyzing a polynomial recurrence satisfied by r_n using techniques from [17].

Since reduced tree pairs are a subset of the set of all tree pairs, it follows that $r_n \leq c_n^2$. We can obtain a lower bound on r_n by the following construction. For every rooted binary tree with $n-1$ carets, T, there is a reduced tree pair of n carets whose left-tree consists of a root caret with T appended to its left leaf, and whose right tree consists of a string of n carets, each appended to the right leaf of its parent. So $r_n > c_{n-1}$ and further $c_n^{1/n} < r_n^{1/n} < c_n^{2/n}$. Since $c_n \sim B4^n n^{-3/2}$ (see [8] for example), it follows that $4 \leq \lim_{n\to\infty} r_n^{1/n} \leq 16$.

The differential equation satisfied by $R(z)$ can be transformed into a linear difference equation satisfied by r_n using the Maple package `GFUN` [15]:

$$\begin{aligned} 0 = &-64n^2(n+1)r_n + 32(6n+5)(n+2)(n+1)r_{n+1} \\ &- 4(n+3)(53n^2+208n+195)r_{n+2} + 2(4n+15)(n+4)(13n+33)r_{n+3} \\ &- (n+5)(n+4)(21n+101)r_{n+4} + (n+5)(n+6)^2 r_{n+5}. \end{aligned}$$

To compute the asymptotic behavior of the solutions of this recurrence we will use the technique described in [17]. This technique has also been automated by the command `Asy()` in the `GuessHolo2` Maple package. This package is available from Doron Zeilberger's website at the time of writing. We outline the method below.

Theorem 1 of [17] implies that the solutions of linear difference equations

$$\sum_{\ell=0}^{\nu} a(n) f_{n+\ell} = 0,$$

where $a(n)$ are polynomials, have a standard asymptotic form. While this general form is quite complicated (and we don't give it here), we note that in the enumeration of combinatorial objects (objects that grow exponentially rather than super-exponentially), one more frequently finds asymptotic expansions of the form

$$f_n \sim \lambda^n n^\theta \sum_{j\geq 0} b_j n^{-j}.$$

By substituting this asymptotic form into the recurrence one can determine the constants λ, θ and b_j. For example, substituting the above form into the recurrence satisfied by r_n, one obtains (after simplifying):

$$0 = (\lambda-1)(\lambda^2-16\lambda+16)(\lambda-2)^2$$
$$+ (\lambda-2)(5\lambda^4\theta+17\lambda^4-256\lambda^3-74\lambda^3\theta+164\lambda^2\theta+558\lambda^2-352\lambda-96\lambda\theta+32)/n$$
$$+ O(1/n^2).$$

In order to cancel the dominant term in this expansion we must have

$$\lambda = 1, 8-4\sqrt{3}, 2, 8+4\sqrt{3}.$$

Each of these value implies different values of θ so as to cancel the second-dominant term. In particular, if $\lambda = 8+4\sqrt{3}$, then $\theta = -3$. Since $4 \leq \lim r_n^{1/n} \leq 16$, it follows that the value of λ which corresponds to the dominant asymptotic growth of r_n must be $8+4\sqrt{3}$.

The application of this process using the full general asymptotic form (thankfully) has been automated by the `GuessHolo2` Maple package. In particular, we have used the `Asy()` command to compute the asymptotic growth of r_n:

$$\frac{n^3}{A\mu^n} r_n \sim 1 + \frac{33/2 - 11\sqrt{3}}{n},$$

for some constant A. □

Though we do not need the exact value of the constant A in our applications below, we can estimate the constant A as follows. Using Stirling's approximation we know that $c_n^2 \sim \frac{1}{\pi n^3} 16^n$. This dictates the behavior of $P(z)$ around its dominant singularity. This then gives the behavior of $R(z)$ around its dominant singularity. Singularity analysis [8] then yields

$$r_n \sim \frac{6-3\sqrt{3}}{\pi n^3}\mu^n \sim \frac{12}{\mu \pi n^3}\mu^n.$$

While this argument is not rigorous, the above form is in extremely close numerical agreement with r_n for $n \leq 1000$.

Proposition 5 (Non-algebraic) *The generating function $R(z)$ is not algebraic.*

Proof: Theorem D of [7] states that if $l(z)$ is an algebraic function which is analytic at the origin then its Taylor coefficients l_n have an asymptotic equivalent of the form

$$l_n \sim A\,\beta^n n^s$$

where $A \in \mathbb{R}$ and $s \notin \{-1,-2,-3,\ldots\}$. Since r_n is not of this form, in particular it has an n^{-3} term, then the generating function $R(z)$ cannot be algebraic. □

The generating function, or "growth series," for the actual word metric in Thompson's group F with respect to the $\{x_0, x_1\}$ generating set, is not known to be algebraic or even D-finite. Burillo [3] and Guba [9] have estimates for the growth but there are significant gaps between the upper and lower bounds which prevent effective asymptotic analysis at this time. Since finding differential equations for generating functions can lead to information about the growth rate of the coefficients, more precise understanding of the growth series for F with respect the standard generating set (or any finite generating set) would be interesting and potentially useful.

In later work we will need the following lemma which follows immediately from the asymptotic formula for r_n.

Lemma 6 (Limits of quotients of r_n) *For any $k \in \mathbb{Z}$*

$$\lim_{n\to\infty} \frac{r_{n-k}}{r_n} = \mu^{-k}.$$

Proof: First multiply the ratio by μ^k and then divide r_n and r_{n-k} by their asymptotic behavior to obtain

$$\mu^k \cdot \frac{r_{n-k}}{r_n} = \frac{r_{n-k}}{A\mu^{n-k}(n-k)^{-3}} \cdot \frac{A\mu^n n^{-3}}{r_n} \cdot \left(\frac{n-k}{n}\right)^3.$$

Each of the three quotients on the right-hand side are strictly positive for all $n > k$ and converge to 1 as $n \to \infty$. Since the limit of their product is the product of their limits we obtain the result. $\square$

Finally, we give a formula for r_n. Woodruff ([18] Theorem 2.8) gave the following formula for the number of reduced tree pair diagrams N_n, where he allows the empty tree pair diagram of size 0 to represent the identity and does not allow a tree pair diagram of size 1:

$$N_n = \sum_{k=1}^{\lceil n/2 \rceil} 2^{n-2k+1} \binom{n-1}{n-2k+1} c_{k-1} \sum_{i=0}^{k} (-1)^i \binom{k}{i} c_{n-i}.$$

Note that there is a typographical error in the expression given at one point in [18] and the above equation is the correct version.

One may readily verify (numerically) that Woodruff's formula and ours (below) agree for $n \geq 2$, however we have not been able to directly demonstrate their equality. We have been able to show (using Maple) that both expressions satisfy the same third-order linear recurrence, which together with the equality of the first few terms is sufficient to prove that the expressions are, in fact, equal.

Lemma 7 (Formula for r_n) *The number of reduced tree pairs with n carets in each tree is given by the formula*

$$r_n = \sum_{k=1}^{n} (-1)^{n-k} \binom{k+1}{n-k} c_k^2$$

Proof: From Proposition 2 we have $R(z) = (1-z)P(z(1-z))$ which expands to

$$\begin{aligned}\sum_{n\geq 1} r_n z^n &= (1-z)\sum_{k\geq 1} c_k^2 z^k (1-z)^k \\ &= \sum_{k\geq 1} c_k^2 z^k (1-z)^{k+1}\end{aligned}$$

which by the binomial theorem

$$\begin{aligned}&= \sum_{k\geq 1} c_k^2 z^k \sum_{j=0}^{k+1} (-1)^j \binom{k+1}{j} z^j \\ &= \sum_{k\geq 1} c_k^2 \sum_{j=0}^{k+1} (-1)^j \binom{k+1}{j} z^{k+j}\end{aligned}$$

Now look at the coefficient of z^n on both sides. For the right side, as k runs from 1 up, we get exactly one term from the second summation, when $j = n-k$. Thus we get

$$r_n = \sum_{k\geq 1} c_k^2 \left((-1)^{n-k} \binom{k+1}{n-k} \right)$$

and since $\binom{k+1}{n-k} = 0$ for $k > n$ we have

$$= \sum_{k=1}^{n} c_k^2 (-1)^{n-k} \binom{k+1}{n-k}$$

$\square$

3 Thompson's group F

Richard Thompson's group F is a widely studied group which has provided examples of and counterexamples to a variety of conjectures in group theory. From the perspective of geometric group theory, F is a fascinating group because it can be studied with multiple methods. Algebraically, F has useful descriptions in terms of families of both finite and infinite sets of generators and relations. Analytically, we describe F as the group of piecewise-linear homeomorphisms of the unit interval, subject to the following conditions: all slopes are powers of two, and all breakpoints have coordinates in the set of dyadic rationals. Combinatorially, the elements of F correspond uniquely to a pairs of reduced finite rooted binary trees, each with the same number of carets. In this paper, we mainly use the combinatorial description of F. We now present a brief introduction to Thompson's group F, and refer the reader to Cannon, Floyd and Parry [4] for more details.

Thompson's group F is infinitely presented as

$$\langle x_0, x_1, \dots | x_i^{-1} x_j x_i = x_{j+1},\ i < j \rangle.$$

With respect to this generating set, group elements have a standard normal form: each $g \in F$ can be written as a product of generators of the form

$$x_{i_1}^{e_1} x_{i_2}^{e_2} \dots x_{i_k}^{e_k} x_{j_l}^{-f_l} x_{j_{l-1}}^{-f_{l-1}} \dots x_{j_1}^{-f_1}$$

where $0 \leq i_1 < i_2 \dots < i_k$ and $0 \leq j_1 < j_2 \dots < j_l$ and e_n and f_m are positive for all n, m. This normal form is unique if we require that when x_i and x_i^{-1} both appear in the normal form, so does either x_{i+1} or x_{i+1}^{-1}.

It is clear from the above presentation that only x_0 and x_1 are necessary to generate the entire group, as x_0^i conjugates x_1 to x_{i+1}. Furthermore, after using conjugation to express higher-index generators in terms of x_0 and x_1, the full infinite family of relations are consequences of two basic relations. This yields the finite presentation

$$\langle x_0, x_1 | [x_0 x_1^{-1}, x_0^{-1} x_1 x_0], [x_0 x_1^{-1}, x_0^{-2} x_1 x_0^2] \rangle$$

for F, where $[a, b]$ denotes the commutator $aba^{-1}b^{-1}$.

Every element of F is uniquely represented by a reduced pair of trees as defined in the previous section. The pair of trees representing the element x_1 is shown in Figure 4. Each of the infinite generators x_i has a similar representative using $i+1$ carets: the first tree consists of a string of $i+1$ carets, each the right child of the previous one, and the second tree consists of a similar string of i carets, and a single interior caret which is the left child of the final right caret.

x_1

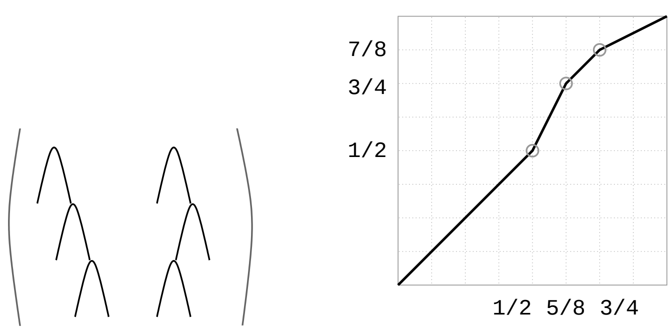

Figure 4: The group element x_1 represented first as a word in the group generators, then as a reduced tree pair, and finally as a piecewise-linear map of the interval $[0, 1]$.

A single finite binary rooted tree of this form determines a subdivision of the unit interval as follows: the root caret divides $[0, 1]$ into the two intervals $[0, \frac{1}{2}]$ and $[\frac{1}{2}, 1]$. The left (respectively right) child of the root subdivides the left (respectively right) interval, and we repeat this subdivision with successive left and right children. We regard carets as corresponding to intervals between dyadic rationals, and we can regard the left and right leaves of a caret as corresponding to the left and right endpoints of those dyadic intervals. Thus a pair of trees determines two subdivisions of the unit interval; we take one for the domain and the other for the range and construct a piecewise-linear homeomorphism using those points as breakpoints. This process is clearly reversible.

A pair (T_1, T_2) of finite rooted binary trees with the same number of carets determines an element of F via the method of *leaf exponents*. We let the number of carets in each tree T_i be n. We number the $n+1$ leaves of each tree from 0 to n beginning with the left-most leaf. The leaf exponent $E(k)$ of the leaf numbered k is the length of the maximal path of left edges beginning at k which does not reach the right side of the tree. For the purposes of notation, we number the leaves in T_1 as $0', 1', 2', \ldots, n'$ and the leaves in T_2 as $0, 1, 2, \ldots, n$. Then the element of F represented by the pair (T_1, T_2) of trees is

$$x_0^{E(0)} x_1^{E(1)} \ldots x_n^{E(n)} x_n^{-E(n')} \ldots x_1^{-E(1')} x_0^{-E(0')}.$$

For example, the pair of trees depicted in Figure 5 corresponds to the element $x_1 x_2 x_1^{-1} \in F$.

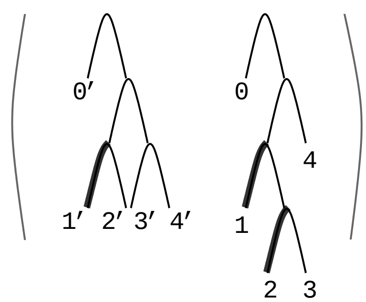

Figure 5: The tree pair diagram representing the group element $x_1 x_2 x_1^{-1}$. The bold edges represent those edges contributing to the leaf exponent of the relevant leaf.

This procedure details a correspondence between elements of F represented as words in the infinite generating set for F and pairs of finite binary rooted trees. An unreduced tree pair will yield a normal form for that element which does not satisfy the uniqueness condition described above, that is, it contains at least one instance of an x_i and x_i^{-1} pair without x_{i+1} or x_{i+1}^{-1}.

3.1 Group multiplication

When elements of F are viewed either algebraically or analytically, the group operation is clear: in the first case it is concatenation and in the second it is composition of functions.

Given a reduced tree pair diagram representing $g \in F$, with leaves numbered from 0 through n, we can create unreduced representatives of g by adding a pair of identical subtrees to leaves with the same number in each tree. If $s = (S_1, S_2)$ and $t = (T_1, T_2)$ are two tree pairs representing elements s and t of F, we obtain the product st as follows. First we create unreduced representatives (S_1', S_2') and (T_1', T_2') of the two elements so that $T_2' = S_1'$. The product st is then given by the tree pair diagram (T_1', S_2'), which may be reduced if necessary. Note that we put the tree pair for s on the right in this operation, the same order as function composition. For example, the product x_1x_2 is shown in Figure 6(d).

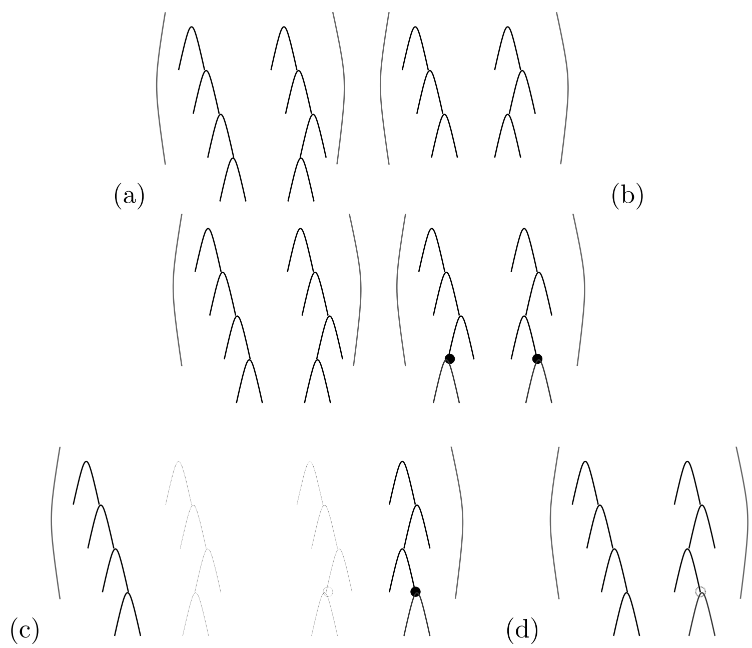

Figure 6: A step-by-step guide to multiplying x_1x_2. Note that we put x_1 on the right, the same as function composition. Figure (a) denotes the two tree pair diagrams, (b) the unreduced representatives needed, (c) the removing of the identical trees, and (d) the final result.

A reduced tree pair (T, S) for an element s is called *absolutely reduced* if in the multiplication of the reduced tree pair for $s^{\pm k}$ by the (T, S) or (S, T) respectively only involves adding carets, with no reductions, for any $k \geq 1$. For example, x_1 in Figure 4 is absolutely reduced.

3.2 Recognizing support and commuting elements

The *support* of an element of F regarded as a homeomorphism of $[0, 1]$ is the closure of the set of points $x \in [0, 1]$ such that $f(x) \neq x$; that is, the set of points which are moved by f. Away from the support of f, the map f will coincide with the identity. From the graph representing an element as a homeomorphism, it is easy to recognize the complement of the full support of an element by inspecting where it coincides with the identity; x_1, for example, has support $[1/2, 1]$ as it coincides with the identity for the first half of the interval. In terms of tree pair diagrams, it is not so easy to recognize the complete support directly from the tree pair diagrams. Nevertheless, it is possible to tell easily if the support extends to the endpoints 0 and 1 of the interval, by inspecting the locations of first and last leaves of the trees S and T representing an element.

If the distances of the leftmost leaves (the leaves numbered 0) in S and T from their respective roots are both k, then the homeomorphism represented by this pair of trees coincides with the identity at least on the interval $[0, \frac{1}{2^k}]$. If there are, in addition to the leaves numbered 0, a sequence of leaves numbered $1, \ldots, m$, each of which have the same distances from the root in both trees, then the homeomorphism will coincide with the identity from 0 to the endpoint of the dyadic interval represented by leaf m. Similarly, near the right endpoint 1, if the distances of the rightmost leaves (those numbered n) in S and T from their respective roots are both l, then the homeomorphism represented coincides with the identity at least on the interval $[1-\frac{1}{2^l}, 1]$. Again, if there are sequences of leaves numbered from $n-m$ up to n which have the same levels in the trees S and T, then the homeomorphism will coincide with the identity on the corresponding dyadic interval, ending at the right endpoint of 1. Elements that have homeomorphisms that coincide with the identity for intervals of positive length at both the left and right endpoints are of particular interest as those elements lie in the commutator subgroup of F, as described below.

Elements of F can commute for many different reasons, but one of the simplest reasons that elements can commute is that they have disjoint supports. By taking elements that have been constructed to have disjoint supports, we can ensure that these elements commute. An illustrative example is simply the construction of a subgroup of F isomorphic to $F \times F$, where the four generators used are pictured in Figure 7. The first two generators have support lying in the interval $[\frac{1}{2}, 1]$ and generate a copy of F with support in that interval. Similarly, the second two generators have support lying in $[0, \frac{1}{2}]$ and generate a commuting copy of F in that interval. We refer to this example as the *standard* $F \times F$ subgroup of F and will make use of it in later sections.

3.3 More subgroups of F

We now describe some subgroups of F which will play key roles in later theorems in this paper. One important subgroup of F is the restricted wreath product $\mathbb{Z} \wr \mathbb{Z}$. Guba and Sapir [10] proved a dichotomy concerning subgroups of F: any subgroup

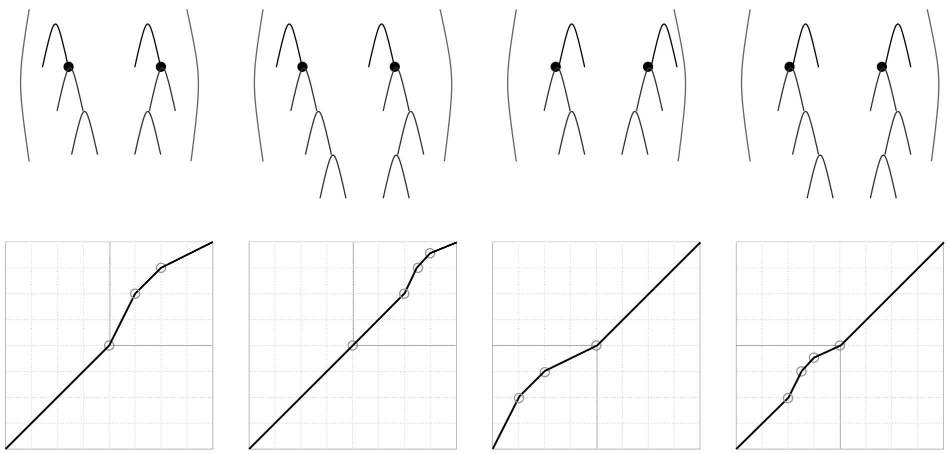

Figure 7: Generators of the standard $F \times F$ subgroup of F.

of F is either free abelian or contains a subgroup isomorphic $\mathbb{Z} \wr \mathbb{Z}$. A representative example of a subgroup of F isomorphic to $\mathbb{Z} \wr \mathbb{Z}$ is easily seen to be generated by the elements x_0 and $y = x_1 x_2 x_1^{-1}$, whose tree pair diagram we gave in Figure 5. The conjugates of y by x_0 have disjoint support and thus commute.

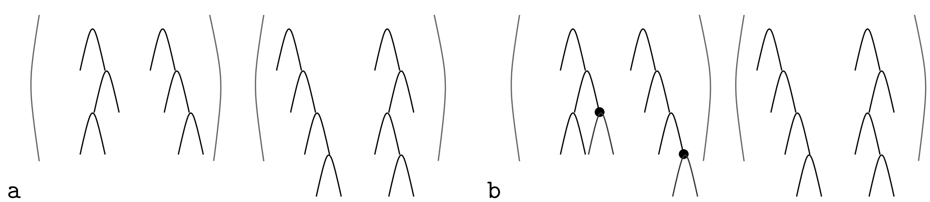

Figure 8: Multiplying $(x_1 x_2) x_1^{-1}$. Deleting the middle trees in (b) we obtain the tree pair in Figure 5. This tree pair together with x_0 generates a $\mathbb{Z} \wr \mathbb{Z}$ subgroup.

Other wreath product subgroups of F include $F \wr \mathbb{Z}$ and $H \wr \mathbb{Z}$ for any $H < F$. Generators for $H \wr \mathbb{Z}$ are obtained as follows. Let $\{h_1, \cdots, h_k\}$ be a generating set for H where $h_i = (T'_i, S'_i)$. Let T be the tree with two right carets, and leaves numbered $1, 2, 3$. Define generators $k_i = (T_i, S_i)$ for $H \wr \mathbb{Z}$ by letting T_i be the tree T with T'_i attached to leaf 2, and S_i be the tree T with S'_i attached to leaf 2. Then $\{k_i\} \cap \{x_0\}$ forms a generating set for $H \wr \mathbb{Z}$.

The group F contains a multitude of subgroups isomorphic to F itself; any two distinct generators from the infinite generating set for F will generate such a subgroup. More generally, Cannon, Floyd and Parry describe a simple arithmetic condition to guarantee that a set of analytic functions of the interval with the appropriate properties generates a subgroup of F which is isomorphic to F.

Lemma 8 ([4] Lemma 4.4) *We let a and b be dyadic rational numbers with $0 \leq a < b \leq 1$ such that $b - a$ is a power of 2. Then the subgroup of F consisting of all functions with support in $[a, b]$ is isomorphic to F by means of the straightforward linear conjugation.*

We can also construct proper subgroups of F which are isomorphic to F combinatorially. A finite string of 0's and 1's gives an address of a vertex in a rooted binary tree with a specified leaf as follows. We start with a root caret and mark its left leaf if the first letter of the string is 0, else mark the right leaf. For each letter, add a caret to the marked leaf and mark its left (respectively right) leaf for a 0 (respectively 1). We let T be a tree constructed in this way with final marked vertex v. Then construct two tree pairs h_0 and h_1 as follows. Denote $x_0 = (T_{x_0}, S_{x_0})$ and $x_1 = (T_{x_1}, S_{x_1})$. Draw four copies of the tree T which we number T_1 through T_4. To the marked vertex in T_1 we attach T_{x_0} and to the marked vertex in T_2 we attach S_{x_0}, forming the tree pair diagram representing h_0. We do the same thing with T_3, T_4, T_{x_1} and S_{x_1} to form h_1. Then h_0 and h_1 generate a subgroup of F isomorphic to F, which is called a *clone subgroup* in [6] and consists of elements whose support lies in the dyadic interval determined by the vertex v. Subgroups of this form are easily seen to be quasi-isometrically embedded. See Figure 9 for an example of two elements which generate a subgroup isomorphic to F.

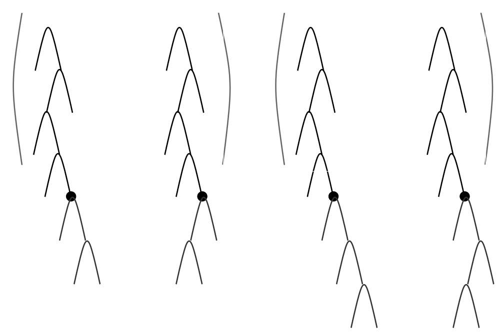

Figure 9: These two elements generate a subgroup isomorphic to F. The address string for this example was 1011.

This geometric idea is easily extended to construct subgroups of F isomorphic to F^n.

Another family of important subgroups of F are the subgroups isomorphic to $\mathbb{Z}^n$, which will play a role in the proofs in Sections 4 and 5. We let T be the tree with $n - 1$ right carets, and n leaves, and (A_i, B_i) for $i = 1, 2, \ldots, n$ reduced pairs of trees all with the same number of carets. We construct generators $h_i = (C_i, D_i)$ of $\mathbb{Z}^n$ as follows. We let C_i be the tree T with A_i attached to leaf i, and D_i the tree T with B_i attached to leaf i, as shown in Figure 10. We reduce the pair (C_i, D_i) if necessary. It is easy to check by multiplying the tree pair diagrams that $h_i h_j = h_j h_i$ for $i, j = 1, 2, \ldots, n$ and thus these elements generate a subgroup

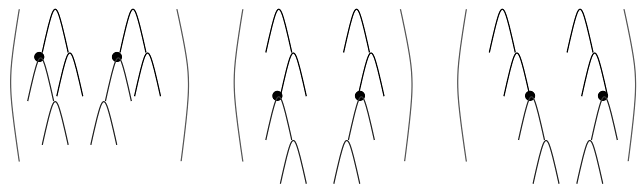

Figure 10: Three tree pairs h_1, h_2, h_3 used to generate $\mathbb{Z}^3$. We have used the tree pair diagram for x_0 as each pair (A_i, B_i). Note that the first pair can be reduced to a tree pair diagram containing only three carets by deleting the rightmost exposed caret.

of F isomorphic to $\mathbb{Z}^n$. Burillo [2] exhibits a different family of subgroups of F isomorphic to $\mathbb{Z}^n$ using the generators $\{x_0x_1^{-1}, x_2x_3^{-1}, x_4x_5^{-1}, \dots x_{2n-2}x_{2n-1}^{-1}\}$ which he shows are quasi-isometrically embedded. In fact, Burillo proves that any infinite cyclic subgroup of F is undistorted, or quasi-isometrically embedded.

3.4 The commutator subgroup of F

In the proofs in Sections 4 and 5 below, we use both algebraic and geometric descriptions of the commutator subgroup $[F, F]$. This subgroup of F has two equivalent descriptions:

- The commutator subgroup of F consists of all elements in F which coincide with the identity map (and thus have slope 1) in neighborhoods both of 0 and of 1. This is proven as Theorem 4.1 of [4].
- The commutator subgroup of F is exactly the kernel of the map $\varphi : F \to \mathbb{Z} \bigoplus \mathbb{Z}$ given by taking the exponent sum of all instances of x_0 in a word representing $w \in F$ as the first coordinate, and the exponent sum of all instances of x_1 as the second coordinate.

The exponent sum homomorphism φ is closely tied to another natural homomorphism ϕ from F to $\mathbb{Z} \bigoplus \mathbb{Z}$. The "slope at the endpoints" homomorphism ϕ for an element $f \in F$ takes the first coordinate of the image to be the logarithm base 2 of the slope of f at the left endpoint 0 of the unit interval and the second coordinate to be the logarithm base 2 of the slope at the right endpoint 1. The images of the generators under the slope-at-the-endpoints homomorphism ϕ are $\phi(x_0) = (1, -1)$ and $\phi(x_1) = (0, -1)$ and it is easy to see that ϕ and φ have the same kernel.

It is not hard to see that the first description above has the following geometric interpretation in terms of tree pair diagrams. An element of the commutator subgroup will have slope 1 at the left and right endpoints and coincide with the identity on intervals of the form $[0, b_0]$ and $[b_1, 1]$ where b_0 and b_1 are, respectively, the first and last points of non-differentiability in $[0, 1]$. These points must lie on the line $y = x$, and the element is represented by tree pair diagrams in which the first leaves (numbered 0) in each tree lie at the same *level* or distance from the root, and the same must be true of the last leaf in each of the trees. Thus, elements of the commutator subgroup are exactly those which have a reduced tree pair diagram (S, T) where the leaves numbered zero are at the same level in both S and T and the last leaves are also at the same level in both S and T. For example, if (A, B) is any reduced n-caret tree pair, then the $(n+2)$-caret tree pair in Figure 11 is also reduced and represents an element in $[F, F]$.

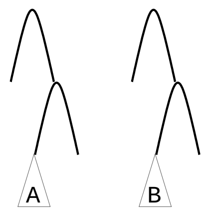

Figure 11: Constructing a tree pair representing a group element which lies in the commutator subgroup $[F, F]$.

We refer the reader to [4] for a proof that the commutator of F is a simple group, and that $F/[F, F] \cong \mathbb{Z} \bigoplus \mathbb{Z}$.

In our arguments below we will be interested in isomorphism classes of subgroups of F. It will sometimes be necessary to assume that a particular finitely generated subgroup of F is not contained in the commutator subgroup $[F, F]$. We now show that within the isomorphism class of any subgroup H of F, it is always possible to pick such a representative. The proof of this lemma follows the proof of Lemma 4.4 of [4].

Lemma 9 (Finding subgroups outside the commutator) *Let H be a finitely generated subgroup of F. Then there is a subgroup H' of F which is isomorphic to H and not contained in the commutator subgroup.*

Proof: If H is not contained in the commutator subgroup $[F, F]$, then take $H' = H$. Otherwise, let H be generated by $h_1, h_2, \ldots, h_k$ where each $h_i \in [F, F]$. Then each h_i has an associated ordered pair (a_i, b_i) where a_i is x-coordinate of the first point of non-differentiability of h_i as a homeomorphism of $[0, 1]$ (necessarily at a_i the slope will change from 1 to something which is not 1.) Similarly, we let b_i be the x-coordinate of the final point of non-differentiability of h_i. We let $a = \min\{a_i\}$ and $b = \max\{b_i\}$. By the choice of a and b, all $h \in H$ have support in $[a, b]$.

Following the proof of Lemma 4.4 of [4], we let $\phi : [a, b] \to [0, b - a]$ be defined by $\phi(x) = x - a$. We use ϕ to define a map on $h \in H$ by $h \mapsto \phi h \phi^{-1}$, assuming that $\phi h \phi^{-1}$ acts as the identity for $x \in (b - a, 1]$. It is clear from the definition of ϕ that the breakpoints of $\phi h \phi^{-1}$ are again dyadic rationals, and the slopes are again powers of two. Since ϕ is an isomorphism, we know that $H \cong \langle \phi h_i \phi^{-1} \rangle$.

But this subgroup cannot be in the commutator, since at least one element, the one which had its minimal breakpoint at $x = a$, now has slope at $x = 0$ which is no longer equal to 1, and thus is not in the commutator subgroup. $\square$

In the proofs in Sections 4 and 5 below, we often want to make a more specific choice of representative subgroup from an isomorphism class of a particular subgroup of F, as follows.

Let $E_i(w)$ for $i = 0, 1$ denote the exponent sum of all instances of x_i in a word w in x_0 and x_1.

Lemma 10 *Let $H = \langle h'_1, h'_2, \ldots, h'_k \rangle$ be a finitely generated subgroup of F. Then there is a subgroup $H' = \langle h_1, h_2, \ldots, h_k \rangle$ isomorphic to H so that $E_0(h_1) \neq 0$ and $E_0(h_j) = 0$ for $j = 2, 3, \ldots, k$.*

Proof: By Lemma 9, we assume without loss of generality that H is not contained in the commutator subgroup $[F, F]$. By replacing some generators with their inverses, we may assume that $E_0(h'_i) \geq 0$ for all i, and that $E_0(h'_1)$ is minimal among those $E_0(h'_i)$ which are positive. For these h'_i, we replace h'_i by $h'_i h_1'^{-d_i}$ where d_i is chosen so that $E_0(h'_i h_1'^{-d_i})$ is as small as possible while non-negative. Repeating this process yields a generating set for a subgroup isomorphic to H with one element having exponent sum on all instances of x_0 equal to zero. We can repeat this process with the remaining generators until a generating set with the desired property is obtained. $\square$

4 Subgroup spectrum with respect to the sum stratification

We now introduce the first of two stratifications of the set of k generator subgroups of Thompson's group F. We view group elements as non-empty reduced tree pairs and denote by X_k the set of unordered k-tuples of non-empty reduced tree pairs $t_i = (T_1^i, T_2^i)$ for $i = 1, \ldots, k$. We denote the number of carets in T_1^i by $|t_i|$. We define the sphere of radius n in X_k as the set of k-tuples having a total of n carets in the k tree pair diagrams in the tuple:

$$\mathrm{Sph}_k^{\mathrm{sum}}(n) = \{(t_1, \ldots, t_k) \mid \sum_{i=1}^{k} |t_i| = n\}$$

which induces a stratification on X_k that we will call the *sum stratification.* Note that we count the number of carets in a tree pair as n when each tree in the tree pair has n carets, while in total the actual number of carets drawn in that tree pair diagram would be $2n$. For example, the triple of tree pairs in Figure 10, once h_1 is reduced, lies in $\mathrm{Sph}_3^{\mathrm{sum}}(11)$.

Recall from Section 1 that the density of a set T of k-tuples of reduced tree pairs is given by

$$\lim_{n\to\infty} \frac{|T \cap \mathrm{Sph}_k^{\mathrm{sum}}(n)|}{|\mathrm{Sph}_k^{\mathrm{sum}}(n)|}$$

with respect to this stratification. If H is a subgroup of F, and T_H the set of k-tuples whose coordinates generate a subgroup of F that is isomorphic to H, recall that H is *visible* if T_H has positive density, and the *k-spectrum* $\mathrm{Spec}_k^{\mathrm{sum}}(F)$ is the set of visible subgroups with respect to the sum stratification of X_k. In this section we explicitly compute these subgroup spectra. We find that any nontrivial subgroup H of F which can be generated by m generators appears in $\mathrm{Spec}_k^{\mathrm{sum}}(F)$ for all $k \geq m$ (Theorem 12). We conclude that this stratification does not distinguish any particular subgroups through the subgroup spectrum, in contrast to the results we will describe in Section 5 when the max stratification is used.

We begin by determining upper and lower bounds on the size of the sphere of radius n in this stratification. The sphere $\mathrm{Sph}_k^{\mathrm{sum}}(n)$ is the set of unordered k-tuples of non-empty reduced tree pairs of sizes $i_1, \ldots, i_k$ with $i_j \geq 1$ so that the sum of the i_j is n. Since they are unordered, we may assume that they are arranged from largest to smallest.

Lemma 11 (Size of $\mathrm{Sph}_k^{\mathrm{sum}}(n)$) *For $k \geq 1$, and $n \geq k$, the size of the sphere of radius n with respect to the sum stratification satisfies the following bounds:*

$$r_{n-k+1} \leq |\mathrm{Sph}_k^{\mathrm{sum}}(n)| \leq r_{n+k-1}.$$

Proof: For the lower bound, $\mathrm{Sph}_k^{\mathrm{sum}}(n)$ contains all k-tuples where the first pair has $n-k+1$ carets and the remaining $(k-1)$ pairs are consist of two single carets. There are r_{n-k+1} ways to choose this first pair, which yields the lower bound.

For the upper bound, we consider the set of all r_{n+k-1} tree pairs with $n + k - 1$ carets in each tree. A (small) subset of these correspond to the k-tuples of $\mathrm{Sph}_k^{\mathrm{sum}}(n)$ as follows. Take the subset of these tree pairs where each tree contains at least $k-1$ right carets, as in Figure 12, where leaf i for $0 \leq i \leq n-1$ has a possibly empty left subtree labeled A_i in T_- and B_i in T_+. Let A_n and B_n respectively denote the right subtrees attached to leaf n in T_- and T_+. The sum of the number of carets in the A_i must equal n.

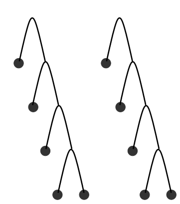

Figure 12: $k-1$ right caret pairs (with $k=5$).

When the number of carets in A_i equals the number of carets in B_i for all i, this pair of trees can be associated to an (ordered) k-tuple of tree pairs with a total of n carets. Amongst these we can find every unordered k-tuple in $\mathrm{Sph}_k^{\mathrm{sum}}(n)$. So this is a gross overcount which suffices to prove the lemma. $\square$

Theorem 12 (All subgroup types are visible with respect to sum) *Let $H = \langle h_1, h_2, \ldots, h_m \rangle$ be a nontrivial subgroup of F. Then $H \in \mathrm{Spec}_k^{\mathrm{sum}}(F)$ for all $k \geq m$.*

We use the notation from Section 3.4 to represent the exponent sum of different generators in a word in x_0 and x_1. Let $E_i(w)$ for $i = 0, 1$ denote the exponent sum of x_i in a group element given by a word w.

Proof: Applying Lemmas 9 and 10, we may assume that H is a representative of its isomorphism class which is not contained in the commutator subgroup $[F, F]$ and such that $E_0(h_1) \neq 0$ but $E_0(h_i) = 0$ for $i > 1$.

We now construct a set of k generators $l_i = (T_i, S_i)$ for $i = 1, 2, \ldots, k$ using a total of n carets which we will show generate a subgroup of F isomorphic to H. We let $h_i = (T_i', S_i')$ as a tree pair diagram, and $s = \sum_{i=1}^m |h_i|$. We let (A, B) be a reduced pair of trees with $n - (s + k)$ carets in each tree. We take n to be larger than $s + k$ in order to construct (A, B) in this way. We define l_1 by taking T_1 to be the tree with a root caret whose left subtree is T_1' and whose right subtree is A. Similarly, we let S_1 be the tree with a root caret whose left subtree is S_1' and whose right subtree is B.

For $2 \leq i \leq m$, we let T_i be the tree consisting of a root caret whose left subtree is T_i' and whose right subtree is empty. We let S_i be the tree consisting of a root caret whose left subtree is S_i' and whose right subtree is empty. For $m+1 \leq i \leq k$, we let l_i be the identity represented by a pair of trees each containing a single caret.

We note that by construction, all tree pair diagrams constructed in this way are reduced, and the total number of carets used (counting just one tree in each pair) in this tuple is $n - (k + s) + s + m + (k - m) = n$ ensuring that this lies in the desired sphere.

It is clear that $\langle l_1, l_2, \ldots, l_k \rangle$ generate a subgroup of $H \times \mathbb{Z}$, where the isomorphic copy of H lies in the first factor of the standard $F \times F$ subgroup of F and where we take (A, B) to be the generator of the $\mathbb{Z}$ factor which lies in the second factor of the standard $F \times F$ subgroup. We now claim that $\langle l_1, l_2, \ldots, l_k \rangle \cong H$. We use the coordinates (h, t^a) on $H \times \mathbb{Z}$, where $h \in H$ and $t = (A, B)$. We define a homomorphism from $H \times \mathbb{Z}$ to H by taking the first coordinate of (h, t^a). When restricted to $\langle l_1, l_2, \ldots, l_k \rangle$, this map is onto by construction.

To show this projection map is injective, we suppose that $(1, t^a)$ lies in the kernel, for $a \neq 0$. Thus $\langle l_1, l_2, \ldots, l_k \rangle \subset H \times \mathbb{Z}$ has a relator ρ which, when projected to H, yields a relator r of H, and when considered as a word in $\langle l_1, l_2, \ldots, l_k \rangle$, has a second coordinate not equal to the identity. But any relator r of H, when each h_1 is written as a word in x_0 and x_1, satisfies $E_0(r) = 0$. Since the only generator of H with $E_0(h_i) \neq 0$ is h_1, we see that r must have the same number of h_1 and h_1^{-1} terms in it. Thus ρ must have the same number of $l_1 = (h_1, t)$ and l_1^{-1} terms.

Since l_1 is the only generator of $\langle l_1, l_2, \ldots, l_k \rangle$ which can change the $\mathbb{Z}$ coordinate of a product, having equal numbers of l_1 and l_1^{-1} terms in our relator ρ implies that when the H coordinate is the identity, the second coordinate must be t^0. Thus projection to the first factor is an isomorphism when restricted to $\langle l_1, l_2, \ldots, l_k \rangle$, and we conclude that this group is isomorphic to H.

We now show that the set of k-tuples of tree pair diagrams constructed in this way is visible in $\mathrm{Spec}_k^{\mathrm{sum}}(F)$. There are $r_{n-(s+k)}$ ways to choose the pair (A, B), which had $n - (s + k)$ carets, and which determined the l_1 generator in this construction. Thus we see that

$$\lim_{n\to\infty} \frac{r_{n-(s+k)}}{|\mathrm{Sph}_k^{\mathrm{sum}}(n)|} \geq \lim_{n\to\infty} \frac{r_{n-(s+k)}}{r_{n+k-1}} = \mu^{-(s-1+2k)} > 0$$

using Lemmas 11 and 6. $\square$

The probabilistic motivation for the definition of a visible subgroup H was that a set of k randomly selected reduced pairs of trees would generate a subgroup isomorphic to H with nonzero probability. In the preceding proof, we were able to show that any given m-generator subgroup is visible in $\mathrm{Spec}_k^{\mathrm{sum}}(F)$ using a k-tuple of pairs of trees consisting of one "large" tree pair diagram, $m-1$ "small" tree pair diagrams, and finally $k - m$ "tiny" tree pair diagrams representing the identity.

For particular subgroups, these estimates of a lower bound on the density of the subgroup are small but positive. It follows from the proof of Theorem 12 that we obtain larger estimates for subgroups with generators which are given by small tree pair diagrams. For example, the asymptotic density of the isomorphism class of the subgroup $\mathbb{Z}$ is at least $\mu^{-5} \approx \frac{1}{750000}$ in the set of all 2-generator subgroups, since $k = 2$ and $\mathbb{Z}$ can be generated by x_0 which has size 2. For other nontrivial subgroups, the construction in this proof will require more carets and the lower bounds we obtain will be even smaller, but always positive.

5 Subgroup spectrum with respect to the max stratification

We now begin to compute the subgroup spectrum with respect to a different stratification, the "max" stratification, of the set of all k-generator subgroups of F. We again let X_k be the set of unordered k-tuples of reduced pairs of trees, and define the sphere of size n to be the collection of k-tuples in which the maximum size of any component is n:

$$\mathrm{Sph}_k^{\mathrm{max}}(n) = \{(t_1, \ldots, t_k) \mid \max_{i\in\{1,2,\ldots,k\}} \{|t_i|\} = n\}$$

For example, the triple of tree pairs in Figure 10 (once h_1 is reduced) lies in $\mathrm{Sph}_3^{\mathrm{max}}(4)$. Defining spheres in this way induces the desired stratification of X_k.

We define the density of a subset $T \subseteq X_n$ with respect to the max stratification by

$$\lim_{n\to\infty} \frac{|T \cap \mathrm{Sph}_k^{\max}(n)|}{|\mathrm{Sph}_k^{\max}(n)|}$$

and $\mathrm{Spec}_k^{\max}(F)$ to be the set of visible isomorphism types of subgroups of F with respect to the max stratification. As we noted at the end of Section 4, the sum stratification is biased towards k-tuples of tree pair diagrams which contain multiple copies of the identity and other "small" pairs of trees having few carets. Using the maximum number of carets in a tree pair diagram to determine size seems to yield a more natural stratification.

We find strikingly different results when we compute $\mathrm{Spec}_k^{\max}(F)$ as compared to $\mathrm{Spec}_k^{\mathrm{sum}}$(F). For example, we show that $\mathbb{Z}$ lies in $\mathrm{Spec}_1^{\max}(F)$ but not in $\mathrm{Spec}_k^{\max}(F)$ for larger values of k.

As in Section 4, we must first obtain bounds on the size of the sphere of radius n in the max stratification. We will use these bounds in the proofs below. We begin with a lemma about sums of r_n.

Lemma 13 (Sums of r_n) *For $n \geq 2$, $\sum_{i=1}^{n-1} r_i \leq r_n$.*

Proof: Since $r_1 = 1 < r_2 = 2$ the statement holds for $n = 2$. We assume for induction the statement is true for $k \geq 2$. Then

$$\sum_{i=1}^{k} r_i = \sum_{i=1}^{k-1} r_i + r_k \leq 2r_k$$

by inductive assumption. We consider the set of reduced tree pairs with $k+1$ carets in each tree, where either the right child of each root is empty, or the left child of each root is empty. In each case there are r_k ways to arrange the k carets on the nonempty leaf, and these tree pairs form disjoint subsets of the set of all reduced pairs of trees with $k+1$ carets. Thus $2r_k \leq r_{k+1}$ which completes the proof. $\square$

Lemma 14 (Size of $\mathrm{Sph}_k^{\max}(n)$) *For $k \geq 1, n \geq k$,*

$$\frac{1}{k!}(r_n)^k \leq |\mathrm{Sph}_k^{\max}(n)| \leq k(r_n)^k$$

Proof: For the lower bound, there are $(r_n)^k$ ordered k-tuples of reduced tree pairs where *every* pair has n carets. Since $\mathrm{Sph}_k^{\max}(n)$ consists of unordered tuples then dividing this by $k!$ gives a lower bound.

For the upper bound, at least one of the k tree pairs must have n carets. For $1 \leq i \leq k$ suppose that i tree pairs have exactly n carets, and the remaining $k-i$ tree pairs have strictly less than n carets. There are at most

$$(r_n)^i$$

(un)ordered i-tuples of n-caret tree pairs, and at most

$$\left(\sum_{j=1}^{n-1} r_j\right)^{k-i}$$

(un)ordered $(k-i)$-tuples of tree pairs with at most $n-1$ carets each.

So for each i the number of unordered k-tuples of tree pairs where i pairs have n carets and $k-i$ pairs have less than n carets is at most

$$(r_n)^i \left(\sum_{j=1}^{n-1} r_j\right)^{k-i} \leq (r_n)^i (r_n)^{k-i} = (r_n)^k$$

by Lemma 13. Since our k-tuples of tree pairs are unordered, without loss of generality we can list the ones containing n carets first.

Thus in total we have at most

$$\sum_{i=1}^{k} (r_n)^k = k(r_n)^k$$

□

We begin by realizing $\mathbb{Z}^k$ in $\mathrm{Spec}_k^{\max}(F)$ for all $k \geq 1$. We prove that $\mathbb{Z} \notin \mathrm{Spec}_k^{\max}(F)$ for $k > 1$, and conjecture that $\mathbb{Z}^m$ is not visible in $\mathrm{Spec}_k^{\max}(F)$ for $k > m$. In the proof below, we construct a particular collection of subgroups of F isomorphic to $\mathbb{Z}^k$, all of whose generators have a common form, and show that this collection of subgroups is visible. Presumably, the actual density of the isomorphism class of subgroups of F isomorphic to $\mathbb{Z}^k$ is considerably larger.

Lemma 15 ($\mathrm{Spec}_k^{\max}(F)$ **is nonempty)** $\mathbb{Z}^k \in \mathrm{Spec}_k^{\max}(F)$ *for all* $k \geq 1$.

Proof: We let T be the tree consisting of a string of $k-1$ right carets. We construct a set of k pairs of trees which generate a subgroup of F isomorphic to $\mathbb{Z}^k$ as described in Section 3.3.

We let (A_i, B_i) be a reduced pair of trees each with $n-(k-1)$ carets for $i = 1, 2, \ldots, k$. We let h_i be the pair of trees obtained by taking the pair (T, T) and attaching A_i to the i-th leaf of the first copy of T, and B_i to the i-th leaf of the second copy of T. We reduce the tree pair generated in this way (which will be necessary for $i = 1, \ldots, k-2$) to obtain the reduced representative for h_i, which we again denote h_i. We note that h_k will have n carets in each tree in its pair, so this tuple does lie in the proper sphere of the stratification. As discussed above, the set $\{h_1, h_2, \ldots, h_k\}$ will generate a subgroup of F isomorphic to $\mathbb{Z}^k$.

We compute the density of the set of k-tuples of pairs of trees constructed in this way to be at least:

$$\lim_{n\to\infty} \frac{(r_{n-k+1})^k}{k(r_n)^k} = \frac{1}{k}\mu^{-k^2+k} > 0$$

using Lemma 6 and the upper bound from Lemma 14. Thus $\mathbb{Z}^k$ is visible in $\mathrm{Spec}_k^{\max}(F)$. $\square$

For example, this shows that the density of $\mathbb{Z}^2$ in the set of 2-generator subgroups is at least $\frac{1}{2}\mu^{-2} \approx \frac{1}{500}$.

We now show that a subgroup H of F cannot appear in $\mathrm{Spec}_k^{\max}(F)$ for values of k smaller than the rank of the abelianization H_{ab}.

Lemma 16 (Abelianization) *We let H be a subgroup of F, and let n be the rank of the abelianization H_{ab} of H. Then $H \notin \mathrm{Spec}_k^{\max}(F)$ for $k < n$.*

Proof: Since the rank of H_{ab} is n, we know that H cannot be generated with fewer than n elements. Thus H cannot be visible in $\mathrm{Spec}_k^{\max}(F)$ for $k < n$. $\square$

Aside from straightforward obstructions like the group rank and the rank of the abelianization, it is not clear what determines the presence of an isomorphism class of subgroup in a given spectrum. In general, it is difficult to show that an isomorphism class of subgroup is not present in a particular spectrum as it is often difficult to systematically describe all possible ways of generating a subgroup isomorphic to a given one. However, in the case of $\mathbb{Z}$, we can show that $\mathbb{Z}$ is not present in the k-spectrum for $k \geq 2$. This highlights a major difference between the composition of $\mathrm{Spec}_k^{\mathrm{sum}}(F)$ and $\mathrm{Spec}_k^{\max}(F)$, since $\mathbb{Z}$ appears in all spectra with respect to the sum stratification but only in the 1-spectrum with respect to the max stratification. Since a subgroup of F with a single generator is either the identity or infinite cyclic, it follows that $\mathrm{Spec}_1^{\max}(F)$ contains only $\mathbb{Z}$.

We will make use of the following fact from Guba and Sapir's development of diagram groups [11], of which F is a prime example.

Lemma 17 (Powers increase carets) *If w is a reduced tree pair with n carets in each tree, then the reduced tree pair for w^{n+1} has strictly more carets than the pair for w^n for $n \geq 0$, and similarly the reduced tree pair for w^{n-1} has strictly more carets than the pair for w^n for $n < 0$.*

Proof: Following [11], we note that a tree pair w is called *absolutely reduced* if when multiplying w^k by w for all $k \geq 1$, no reductions occur (and similarly for w^{-k} by w^{-1}). By Lemmas 15.10, 15.14, 15.5 and their proofs [11], we see that every element is conjugate to an absolutely reduced element, and that conjugates of absolutely reduced elements have at least as many carets than the absolutely reduced element. By Lemma 15.24 and Theorem 15.25 of [11], we see that the number of carets in the kth power of any reduced element with $n = a + b$ carets is of the form $a + b|k|$, for b at least 1, giving the desired result. $\square$

Now we show $\mathbb{Z}$ is not visible in the higher-rank spectrum of F by enumerating the number of ways that 2-generator subgroups isomorphic to $\mathbb{Z}$ can occur.

Theorem 18 ($\mathbb{Z}$ not visible) *With respect to the max stratification, the spectrum $\mathrm{Spec}_1^{\max}(F) = \{\mathbb{Z}\}$ and for any $k \geq 2$, we have that $\mathbb{Z} \notin \mathrm{Spec}_k^{\max}(F)$.*

Proof: It follows from Lemma 15 that $\mathbb{Z} \in \mathrm{Spec}_1^{\max}(F)$. The only other possible candidate for a subgroup isomorphism class in $\mathrm{Spec}_1^{\max}(F)$ is that of the identity, and the only reduced tree pair diagram representing the identity is of size 1. The number of reduced tree pairs representing the identity is 0 for size $n > 1$, and thus the density of the isomorphism class of the identity subgroup is 0. We conclude that $\mathrm{Spec}_1^{\max}(F) = \{\mathbb{Z}\}$.

To see that $\mathbb{Z} \notin \mathrm{Spec}_k^{\max}(F)$ for any $k \geq 2$, we begin by showing that $\mathbb{Z} \notin \mathrm{Spec}_2^{\max}(F)$, and analogous arguments will show that $\mathbb{Z} \notin \mathrm{Spec}_k^{\max}(F)$ for $k > 2$.

We suppose that h_1, h_2 are two tree pairs of size at most n which generate a cyclic group, with at least one tree pair of size n. Then there must be some element h so that $h_1 = h^k$ and $h_2 = h^l$ for some integers k, l. By Lemma 17 the reduced tree pair for h has at most n carets, and for each such h we can take k, l to be any powers of absolute value less than n. So for each h we have at most $(2n)^2$ choices for k, l, so there are at most $4n^2 \sum_{i=1}^{n} r_i \leq 4n^2 r_{n+1}$ by Lemma 13. Using the lower bound for the size of $\mathrm{Sph}_k^{\max}(n)$ from Lemma 14 we obtain an upper bound for the density of all subgroups isomorphic to $\mathbb{Z}$ in $\mathrm{Spec}_2^{\max}(F)$ of

$$\lim_{n\to\infty} \frac{4n^2 r_{n+1}}{\frac{1}{2!}(r_n)^2} = 8 \lim_{n\to\infty} \frac{r_{n+1}}{r_n} \lim_{n\to\infty} \frac{n^2}{r_n} = 8\mu \lim_{n\to\infty} \frac{n^2}{r_n} = 0$$

using Proposition 4 and Lemma 6. □

We note that this approach does not appear to generalize to show that $\mathbb{Z}^m$ is not visible in $\mathrm{Spec}_k^{\max}(F)$ for $k > m$, as it is difficult to recognize when a collection of tree pair diagrams generates a subgroup isomorphic to $\mathbb{Z}^m$ for $m \geq 2$. Furthermore, it seems quite difficult to compute the complete list of subgroups of a given group which appear in $\mathrm{Spec}_k^{\max}(F)$. Indeed, ignoring any consideration of densities, a complete list of even the 2-generated subgroups of F is not known (see [5] Problem 2.4). For $k = 2$ we can say the following.

Proposition 19 (2-spectrum) *Let $H = \langle h_1, h_2\rangle$ be a subgroup of F. Then either H or $H \times \mathbb{Z}$ lies in $\mathrm{Spec}_2^{\max}(F)$. If $H_{ab} \cong \mathbb{Z} \bigoplus \mathbb{Z}$, then $H \in \mathrm{Spec}_2^{\max}(F)$, otherwise $H \times \mathbb{Z} \in \mathrm{Spec}_2^{\max}(F)$.*

Proof: We may assume, quoting Lemmas 9 and 10 that if $H = \langle h_1, h_2\rangle$ that

- $h_1 \notin [F, F]$
- when h_1 is expressed as a word in x_0 and x_1, the exponent sum of all the instances of x_0 is not equal to 0, and
- when h_2 is expressed as a word in x_0 and x_1, the exponent sum of all the instances of x_0 is equal to 0.

As tree pair diagrams, we use the notation $h_i = (S_i, T_i)$.

We create a new set of generators $k_1 = (X_1, Y_1)$ and $k_2 = (X_2, Y_2)$ for a two generator subgroup of F as follows. We let T be the tree consisting entirely of two right carets, whose leaves are numbered $1, 2$ and 3, and let (A, B) and (C, D) be arbitrary reduced pairs of trees so that (A, B) has $n - N(h_1) - 2$ carets in each tree and (C, D) has $n - N(h_2) - 2$ carets in each tree. We construct X_1 by attaching S_1 to leaf 1 of T and A to leaf 2 of T. We construct Y_1 by attaching T_1 to leaf 1 of T and B to leaf 2 of T. We construct X_2 by attaching S_2 to leaf 1 of T and C to leaf 3 of T. We construct Y_2 by attaching T_2 to leaf 1 of T and D to leaf 3 of T, as in Figure 13. Note that each tree has size n, and we assume without loss of generality that $n > \max\{N(h_i)\} + 2$.

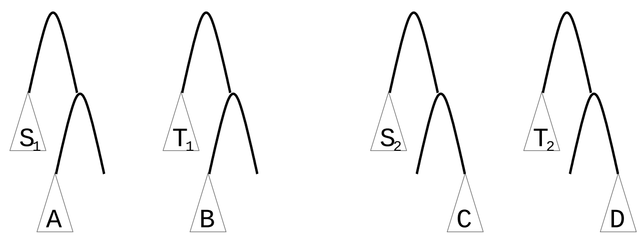

Figure 13: Constructing the tree pairs k_1, k_2 which generate a subgroup of $H \times \mathbb{Z}$.

It is easy to see that k_1 and k_2 generate a subgroup of the standard $F \times F$ subgroup in which the subgroup you obtain on the first factor of F is simply H. Also, $t = (A, B)$ and $s = (C, D)$ each generate a copy of $\mathbb{Z}$ in the second factor of $F \times F$ provided that neither tree pair diagram represents the identity. Let $K \cong \langle k_1, k_2 \rangle$. Then by construction, $K \subset H \times \mathbb{Z}^2$, where the first $\mathbb{Z}$ is generated by $t = (A, B)$ and the second by $s = (C, D)$.

We first show that the set of subgroups K constructed in this way is visible in $\mathrm{Spec}_2^{\max}(F)$, and then we discuss of what isomorphism class of subgroups we have constructed using these elements. By Lemmas 14 and 6 the density of pairs of tree pair diagrams constructed in this way is at least

$$\lim_{n \to \infty} \frac{\left(r_{n-N(h_1)-2}\right)\left(r_{n-N(h_2)-2}\right)}{2(r_n)^2} = \frac{1}{2}\mu^{-N(h_1)-N(h_2)-4} > 0.$$

We claim that K is either isomorphic to H or to $H \times \mathbb{Z}$. Use the coordinates (w, t^a, s^b) on $H \times \mathbb{Z}^2$ where $w \in H$. It is easy to see that for every element $h \in H$, there is at least one $k \in K$ represented by the coordinates (h, t^a, s^b) for some $a, b \in \mathbb{Z}$. We first show that for each $h \in H$, there is a unique second coordinate. Suppose that (h, t^a, s^b) and (h, t^c, s^d) both lie in K, and thus their product (Id, t^{a-c}, s^{b-d}) also lies in K. Thus there is some relation ρ in H expressed in terms of h_1 and h_2 so that when we replace h_i with k_i we obtain the element $(Id, t^{a-c}, s^{b-d}) \in K$. Since the generator t of $\mathbb{Z}$ is linked to h_1 in k_1, and the t coordinate of (Id, t^{a-c}, s^{b-d}) is not zero, we conclude that in ρ, the exponent sum of all instances of the generator h_1 is not equal to zero.

Recall that h_1 was chosen so that when h_1 is expressed as a word in x_0 and x_1, the exponent sum of all the instances of x_0 is not equal to 0, but h_2 does not have this property. Any relation in H can be written in terms of x_0 and x_1 to yield a relation of F, and thus any relation in H must have the total exponent sum of all instances of x_0 equal to 0. By our choice of h_1 and h_2, we see that a relation of H *must* have the exponent sum of all instances of the generator h_1 equal to zero. Thus we must have $a = c$ in our coordinates above.

We have now shown that either $K \cong H \times \mathbb{Z}$ or $K \cong H$. Suppose that $H_{ab} \cong \mathbb{Z} \bigoplus \mathbb{Z}$. Then $(H \times \mathbb{Z})_{ab} \cong \mathbb{Z}^3$ and it follows from Lemma 16 that $H \times \mathbb{Z} \notin \mathrm{Spec}_2^{\max}(F)$. In this case we must have $K \cong H$.

Suppose that $H_{ab} = \mathbb{Z}$. In this case, either $h_2 \in [H, H]$ or $h_1^c = h_2^d w$ for some non-identity element $w \in [H, H]$ and integers c, d. In either case, there is a relator of H in which the total exponent sum on the instances of h_2 is nonzero. Since the s coordinate of the second $\mathbb{Z}$ factor in $H \times \mathbb{Z}^2$ is linked to the h_2 generator in k_2, there is a way to realize both (h, t^a, s^b) and (h, t^a, s^d) in K with $d \neq b$. Thus we must have $K \cong H \times \mathbb{Z}$. $\square$

It follows from Proposition 19 that $\mathrm{Spec}_2^{\max}(F)$ contains $\mathbb{Z}^2, F$, and $\mathbb{Z} \wr \mathbb{Z}$, and from Theorem 18 that it does not contain $\mathbb{Z}$ or $\{Id\}$.

We have seen above that it is difficult to ascertain when a particular isomorphism class of subgroup is present in a given spectrum. Furthermore, the example of $\mathbb{Z}$ shows that presence in a given spectrum does not necessarily imply presence in spectra of higher index.

We find that F is a very special two generator subgroup of itself, and exhibits behavior unlike that of $\mathbb{Z}$. As long as $k \geq 2$, we can show that $F \in \mathrm{Spec}_k^{\max}(F)$. We call this behavior *persistence*; that is, a subgroup H is *persistent* if there is an l so that $H \in \mathrm{Spec}_k^{\max}(F)$ for all $k \geq l$. In the small set of groups whose spectra have been previously studied, no subgroups have shown this persistent behavior. As noted in the introduction, the current known examples of subgroup spectra all find that the free group F_k is generic in the k-spectrum. In Thompson's group F, we find a wealth of examples of this persistent behavior. In the previous section, we effectively proved that *every* non-trivial finitely generated subgroup of F is persistent with respect to the *sum* stratification (Theorem 12). As a corollary of Theorem 20 below and the techniques in Lemma 15 above, it will follow that $F^n \times \mathbb{Z}^m$ and $F^n \wr \mathbb{Z}^m$ are also persistent with respect to the *max* stratification, with $l = 2n + m$.

Theorem 20 (F is persistent) *F lies in $\mathrm{Spec}_k^{\max}(F)$ for all $k \geq 2$.*

Proof: Since F can be generated by two elements, and $F_{ab} \cong \mathbb{Z} \bigoplus \mathbb{Z}$, it follows from Proposition 19 that $F \in \mathrm{Spec}_2^{\max}(F)$. We now show that $F \in \mathrm{Spec}_k^{\max}(F)$ for all $k > 2$.

We define k generators $h_1, h_2, \ldots, h_k$ which generate a subgroup of F isomorphic to F, in such a way that the set of k-tuples pairs of trees of this form is visible. As reduced tree pair diagrams, we use the notation $h_i = (T_i, S_i)$. We begin by defining h_1 and h_2. We let $x_0 = (T_{x_0}, S_{x_0})$ and $x_1 = (T_{x_1}, S_{x_1})$ as tree pair diagrams, (C_1, D_1) any reduced pair of trees with $n - 4$ carets in each tree and (C_2, D_2) any reduced pair of trees with $n-5$ carets in each tree. We let T be the tree with two right carets, and three leaves numbered $1, 2, 3$. We construct h_1 and h_2 as follows:

- We let T_1 be the tree T with T_{x_0} attached to leaf 1 and C_1 attached to leaf 2.
- We let S_1 be the tree T with S_{x_0} attached to leaf 1 and D_1 attached to leaf 2.
- We let T_2 be the tree T with T_{x_1} attached to leaf 1 and C_2 attached to leaf 3.
- We let S_2 be the tree T with S_{x_1} attached to leaf 1 and D_2 attached to leaf 3.

This construction is shown in Figure 14.

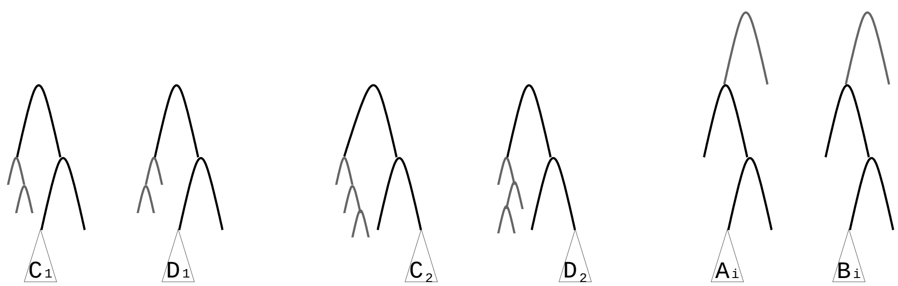

Figure 14: Constructing the tree pairs h_1, h_2, and h_i generating a subgroup of $F \times \mathbb{Z}^2$.

For fixed n, let (A_i, B_i) be any reduced pair of trees with $n-3$ carets for $i = 3, 4, \ldots, k$. Note that there are r_{n-3} ways to choose each such pair. Construct a reduced $(n-1)$-caret tree pair that represents an element of $[F, F]$ by attaching the pair (A_i, B_i) to a 2-caret tree as in Figure 11 in Section 3.4. Call this pair (A_i', B_i'). We now define $h_i = (T_i, S_i)$ for $i = 3, 4, \ldots, k$ as follows:

- let T_i consist of a root caret with A_i' attached to its left leaf, and
- let S_i consist of a root caret with B_i' attached to its left leaf.

The subgroup generated by the $\{h_i\}$ is clearly a subgroup of $F \times \mathbb{Z}^2$, since the subtrees of the h_i which are the left children of the root carets, when taken as independent tree pair diagrams, clearly generate a subgroup H which is isomorphic to F.

Any relator which is introduced into H by the inclusion of the commutators (A_i', B_i') as generators must hold true in F as well. Since all relators of F are commutators or conjugates of commutators, all relators have exponent sum on all instances of either x_0 and x_1 equal to zero. Additionally, we know that x_0 and x_1 are not commutators themselves. Thus any new relators introduced into H by the inclusion of the commutators (A'_i, B'_i) as generators must also have exponent sum on all instances of either x_0 and x_1 equal to zero. Using the coordinates (w, t^a, s^b) for elements of H, where $w \in F$, $t = (C_1, D_1)$ and $s = (C_2, D_2)$, the argument given in Proposition 19 goes through exactly to show that $w \in F$ has unique second and third coordinates, and thus $H \cong F$.

To see that the set of k-tuples constructed in this way is visible, note that the number of ways to construct them is $r_{n-4}r_{n-5}(r_{n-3})^{k-2}$. The choices are in the C_i, D_i trees which generate $\mathbb{Z}^2$, and the A'_i, B'_i trees which are used to construct elements of $[F, F]$. Thus we compute the density of this set of k-tuples to be at least

$$\lim_{n\to\infty} \frac{r_{n-4}r_{n-5}(r_{n-3})^{k-2}}{k(r_n)^k} = \frac{1}{k}\mu^{-4}\mu^{-5}(\mu^{-3})^{k-2} > 0$$

by Lemmas 14 and 6. □

This proof used two very special properties of the whole group F which are not generally true for subgroups of F. First, there is an explicit way of characterizing tree pair diagrams corresponding to elements in the commutator subgroup $[F, F]$, which allows us to construct commutators containing a large arbitrary tree. Second, the relators of F are all commutators themselves, and thus including additional commutators as generators yields relators with the appropriate exponent sums on x_0 and x_1. Thus we do not expect this persistent behavior from other randomly chosen subgroups of F. However, we can adapt the ideas used above to prove that if a subgroup H of F is visible in a particular spectrum, $\mathrm{Spec}_k^{\max}(F)$, then both the wreath product and the cross product with $\mathbb{Z}$ are visible in $\mathrm{Spec}_{k+1}^{\max}(F)$. As a corollary of this fact and Theorem 20, we find that subgroups which contain F as a factor are indeed persistent. We first need the following straightforward lemma about densities of visible subgroups.

Lemma 21 *We let $H_k(n)$ denote the set of all k-tuples of tree pair diagrams which generate a subgroup of F isomorphic to H with a maximum of n carets in any pair of trees, such that at least one coordinate realizes this maximum. If a subgroup H is visible in* $\mathrm{Spec}_k^{\max}(F)$ *then*

$$\lim_{n\to\infty} \frac{|H_k(n)|}{(r_n)^k} \geq \lambda_k$$

for some $\lambda_k \in (0, 1]$.

Proof:

$$\lim_{n\to\infty} \frac{|H_k(n)|}{(r_n)^k} \geq \lim_{n\to\infty} \frac{|H_k(n)|}{k!|\mathrm{Sph}_k^{\max}(n)|}$$

by Lemma 14. Since H is visible this limit equals the density of H with respect to the max stratification, and is positive, which gives the result. $\square$

Proposition 22 (Closure under products) *If* $H \in \mathrm{Spec}_k^{\max}(F)$ *then* $H \times \mathbb{Z}$ *and* $H \wr \mathbb{Z}$ *lie in* $\mathrm{Spec}_{k+1}^{\max}(F)$.

Proof: We construct the $k+1$ generators necessary to obtain a family of subgroups of F isomorphic to $H \times \mathbb{Z}$ in such a way that the set of $(k+1)$-tuples of this form is visible. The techniques are similar to those used above.

We let $h_1, h_2, \ldots h_k$ be a set of k generators for H. We will construct a set $l_1, l_2, \ldots, l_{k+1}$ of generators for $H \times \mathbb{Z}$. We let $h_i = (T_i', S_i')$ as a reduced pair of trees, and we must define $l_i = (T_i, S_i)$. For $i = 1, \ldots, k$ we let T_i consist of a root caret with T_i' as its left subtree, and S_i consist of a root caret with S_i' as its left subtree. We let (A, B) be a reduced pair of trees with $n-1$ carets. To define l_{k+1}, let T_{k+1} consist of a root caret with A as its right subtree, and S_{k+1} consist of a root caret with B as its right subtree.

It is clear that the set $\{l_i\}$ generate a subgroup of F isomorphic to $H \times \mathbb{Z}$. We now show that the set of $(k+1)$-tuples constructed in this way is visible in $\mathrm{Spec}_{k+1}^{\max}(F)$.

To compute the density of the set of $(k+1)$-tuples constructed in this way which generate a subgroup of F isomorphic to $H \times \mathbb{Z}$, we compute the following limit.

$$\lim_{n\to\infty} \frac{|H_k(n-1)| r_{n-1}}{|\mathrm{Sph}_{k+1}^{\max}(n)|} \geq \lim_{n\to\infty} \frac{|H_k(n-1)| r_{n-1}}{(k+1)(r_n)^{k+1}}$$

by Lemma 14

$$\begin{aligned}
&= \lim_{n\to\infty} \frac{1}{k+1} \frac{|H_k(n-1)|}{(r_n)^k} \frac{r_{n-1}}{r_n} \\
&= \lim_{n\to\infty} \frac{1}{k+1} \frac{|H_k(n-1)|}{(r_{n-1})^k} \frac{(r_{n-1})^k}{(r_n)^k} \frac{r_{n-1}}{r_n} \\
&\geq \frac{\lambda_k \mu^{-k-1}}{k+1} > 0.
\end{aligned}$$

by Lemmas 21 and 6.

To see that $H \wr \mathbb{Z}$ lies in $\mathrm{Spec}_{k+1}^{\max}(F)$ under the same assumption on H, we construct slightly different generators, and make an argument analogous to that in Theorem 20. As above, we let $h_1, h_2, \ldots h_k$ be a set of k generators for H. We will construct a set $l_1, l_2, \ldots, l_{k+1}$ of generators which will generate a subgroup of $(H \wr \mathbb{Z}) \times \mathbb{Z}$ which we show to be isomorphic to $H \wr \mathbb{Z}$.

Let $H_k(n-3)$ be the set of all k-tuples which generate a subgroup of F isomorphic to H, where at least one tree pair contains $n-3$ carets. Let $\{h_i = (T_i', S_i')\} \in H_k(n-3)$. Since H is visible in $\mathrm{Spec}_k^{\max}(F)$, Lemma 21 implies that

$$\lim_{n\to\infty} \frac{|H_k(n-3)|}{r_{n-3}^k} > 0.$$

We define $l_i = (T_i, S_i)$ for $i = 1, 2, \ldots, k+1$ as follows. We let T be the tree with two left carets, and one interior caret attached to the right leaf of the caret which is not the root. Number the leaves of T by $1, 2, 3, 4$. For $i = 1, 2, \ldots, k$, let T_i be the tree T with T_i' attached to leaf 2. We let S_i be the tree T with S_i' attached to leaf 2. We let (A, B) be any reduced pair of trees with $n-3$ carets. We let $x_0 = (T_{x_0}, S_{x_0})$. We define l_{k+1} by taking T_{k+1} to be a single root caret with T_{x_0} attached to its left leaf and A attached to its right leaf. We let S_{k+1} be a single root caret with S_{x_0} attached to its left leaf and B attached to its right leaf. See Figure 15.

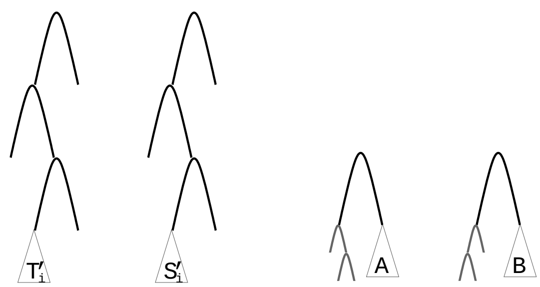

Figure 15: Constructing the tree pairs l_i and l_{k+1} generating a subgroup of $(H \wr \mathbb{Z}) \times \mathbb{Z}$.

It is clear by the construction of our generators that any element of $H \wr \mathbb{Z}$ can appear as the pair of left subtrees of the root carets in any element of $\langle h_i \rangle$. However, we must show that $\langle h_i \rangle$ generates a subgroup of F isomorphic to $H \wr \mathbb{Z}$ and not $(H \wr \mathbb{Z}) \times \mathbb{Z}$. To do this, we note that any since $H \wr \mathbb{Z}$ is a wreath product, all relators are commutators. Thus the argument in Theorem 20 can be applied to show that $\langle h_i \rangle \cong H \wr \mathbb{Z}$ rather than $(H \wr \mathbb{Z}) \times \mathbb{Z}$.

We must now show that the set of $(k+1)$-tuples generated in this way is visible in $\mathrm{Spec}_{k+1}^{\max}(F)$. We let $H_k(n)$ be the set of all k-tuples of tree pair diagrams which generate a subgroup of F isomorphic to H with a maximum of n carets in any pair of trees, such that at least one coordinate realizes this maximum. The density of the set of $(k+1)$-tuples constructed in this way which generate a subgroup of F isomorphic to $H \wr \mathbb{Z}$ is computed as follows. We have r_{n-3} choices for the pair (A, B), and $|H(n-3)|$ is the number of (T_i', S_i') generating sets for H with a maximum of $n-3$ carets in some pair. So together the density is

$$
\begin{aligned}
\lim_{n\to\infty} \frac{|H(n-3)|r_{n-3}}{|\mathrm{Sph}^{\max}_{k+1}(n)|} &\geq \lim_{n\to\infty} \frac{1}{k+1} \frac{|H(n-3)|r_{n-3}}{(r_n)^k r_n} \\
&\geq \lim_{n\to\infty} \frac{1}{k+1} \frac{|H(n-3)|}{(r_{n-3})^k} \frac{(r_{n-3})^k}{(r_n)^k} \frac{r_{n-3}}{r_n} \\
&= \frac{1}{k+1} \lambda_k \mu^{-3k-3} > 0
\end{aligned}
$$

by Lemmas 21 and 6.

$\square$

This proposition allows us to find many subgroups types in k-spectra of the appropriate ranks.

Corollary 23 (Iterated wreath products are in the spectrum) *The l-fold iterated wreath product of $\mathbb{Z}$ with itself $\mathbb{Z} \wr \cdots \wr \mathbb{Z}$ lies in $\mathrm{Spec}^{\max}_l(F)$.*

We note as a further corollary of Proposition 22 that persistence of subgroups is preserved under direct and wreath products with $\mathbb{Z}$.

Corollary 24 (Persistence under direct and wreath products) *If H is a persistent subgroup present in $\mathrm{Spec}^{\max}_k(F)$ for $k \geq l$, then $H \times \mathbb{Z}$ and $H \wr \mathbb{Z}$ are persistent subgroups present in $\mathrm{Spec}^{\max}_k(F)$ for $k \geq l+1$.*

The next two corollaries follow from Theorem 20 and Proposition 22.

Corollary 25 ($F^n \times \mathbb{Z}^m$ is persistent) *For $n \geq 1, m \geq 0$, and for all $k \geq 2n + m$, we have that $F^n \times \mathbb{Z}^m$ lies in $\mathrm{Spec}^{\max}_k(F)$.*

Corollary 25 shows that it is possible to have a subgroup H of F so that both H and $H \times \mathbb{Z}$ are contained in the $\mathrm{Spec}^{\max}_k(F)$ for the same value of k; we can take $H = F^n \times \mathbb{Z}^m$ and $k > 2m + n$. The analogous result is true for wreath products of F^n and $\mathbb{Z}$ as well.

Corollary 26 ($F^n \wr \mathbb{Z}$ is persistent) *$F^n \wr \mathbb{Z}$ lies in $\mathrm{Spec}^{\max}_k(F)$ for $n \geq 1$ for all $k \geq 2n + 1$.*

More generally, we can see that persistent subgroups can "absorb" other visible subgroups present to form new persistent subgroups.

Theorem 27 (Products with persistent subgroups are persistent) *If H is a subgroup which is present in $\mathrm{Spec}^{\max}_k(F)$ and K is a persistent subgroup which is present in $\mathrm{Spec}^{\max}_l(F)$ for $l \geq l_0$, then $H \times K$ is persistent and present in $\mathrm{Spec}^{\max}_l(F)$ for $l \geq l_0 + k$.*

Proof: Let $H_k(n)$ denote the set of all k-tuples of tree pair diagrams which generate a subgroup of F isomorphic to H with a realized maximum of n carets in some coordinate. Since $H \in \mathrm{Spec}_k^{\max}(F)$ we know from Lemma 21 that

$$lim_{n\to\infty}\frac{|H_k(n)|}{(r_n)^k} \geq \lambda_k$$

for some $\lambda_k \in (0, 1]$.

Let $K_l(n)$ denote the set of all l-tuples of tree pair diagrams which generate a subgroup of F isomorphic to K with a realized maximum of n carets in some coordinate. Since K is persistent, we know that for any $l \geq l_0$, the limit

$$lim_{n\to\infty}\frac{|K_l(n)|}{(r_n)^l} \geq \lambda_l$$

for some $\lambda_l \in (0, 1]$.

Let $m = k + l$ for any $l \geq l_0$. Form a generating set $\{t_1, t_2, \cdots, t_m\}$, where $t_i = (T_i, S_i)$, for $H \times K$ as follows. Take any k-tuple $\delta \in H_k(n)$, where $\delta_i \in \delta$ is represented by the pair of trees (T_i^δ, S_i^δ). Take any l-tuple $\eta \in K_l(n)$, where $\eta_j \in \eta$ is represented by the pair of trees (T_j^η, S_j^η).

1. For $1 \leq i \leq k$, let T_i consist of a root caret with left subtree T_i^δ.
2. For $1 \leq i \leq k$, let S_i consist of a root caret with left subtree S_i^δ.
3. For $k + 1 \leq i \leq m$, let T_i consist of a root caret with right subtree T_i^η.
4. For $k + 1 \leq i \leq m$, let S_i consist of a root caret with right subtree S_i^η.

It is clear that this set of tree pairs generates a subgroup of $F \times F$ isomorphic to $H \times K$. A lower bound on the density of the isomorphism class of $H \times K$ is given by the following positive valued limit:

$$\lim_{n\to\infty}\frac{|H_k(n)||K_l(n)|}{r_n^{k+l}} = \lim_{n\to\infty}\frac{|H_k(n)|}{r_n^k}\frac{|K_l(n)|}{r_n^l} \geq \lambda_k\lambda_l > 0.$$

□

Thus, our analysis shows that the following subgroups are present in the k-spectrum with respect to the max stratification:

- The persistent subgroups F, $F \times F$, $\ldots F^n$ for $2n \leq k$.
- The persistent subgroups $F^n \times \mathbb{Z}^m$, for $2n + m \leq k, n \geq 1$.
- The persistent subgroups $F^n \wr \mathbb{Z}$ for $2n + 1 \leq k, n \geq 1$.
- The abelian subgroup $\mathbb{Z}^k$ and the k-fold iterated product of $\mathbb{Z}$ with itself.
- The mixed direct and wreath products of $\mathbb{Z}$ with itself with k terms, including for example $\mathbb{Z}^{k-1} \wr \mathbb{Z}$ and $(\mathbb{Z} \wr \mathbb{Z} \wr \mathbb{Z}) \times \mathbb{Z}^{k-3}$.

- Various mixed direct and wreath products with $\mathbb{Z}$ such as $(F^2 \times \mathbb{Z}^3) \wr \mathbb{Z} \times \mathbb{Z}$ which is present in all $k \geq 9$, for example.

We note that though there are positive densities for each of these isomorphism classes of subgroup, the lower bounds we obtain for their densities are very small and the lower bound on their sum is still tiny, amounting to much less than 1% in all cases where $k > 1$.

We close with some open questions. There are two supergroups of F, Thompson's groups T and V, which are related to F. Both T and V contain free subgroups of rank > 1, unlike F. So it is natural to ask if a random k generated subgroup of either of these groups will be free of rank k with probability 1. At present F is the only group known to have non-generic subgroups, and in all other cases the generic subgroup is always free of the correct rank. More generally, is it the case that if a group has free subgroups of rank > 1 that free groups dominate the subgroup spectrum?

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