1. **10 marks** Let $X_n$ be a Markov chain on $S = \{0, 1, 2\}$ with transition matrix

$$
P = \begin{pmatrix}
\frac{1}{3} & \frac{2}{3} & 0 \\
\frac{1}{2} & \frac{1}{4} & \frac{1}{4} \\
1 & 0 & 0 
\end{pmatrix}.
$$

(a) Find the 2-step transition matrix of $X_n$.
(b) Suppose $X_0 = 0$. What is the mean number of steps until $X_n$ visits 2?
(c) What is the stationary distribution?

**Solution:**

(a) This is $P^2 = \begin{pmatrix}
\frac{4}{9} & \frac{7}{18} & \frac{1}{6} \\
\frac{7}{21} & \frac{19}{48} & \frac{1}{16} \\
\frac{1}{3} & \frac{2}{3} & 0 
\end{pmatrix}$

(b) Let $m_i$ be the mean from state $i$. Then

$$m_0 = 1 + \frac{1}{3}m_0 + \frac{2}{3}m_1 \quad m_1 = 1 + \frac{1}{2}m_0 + \frac{1}{4}m_1.$$ 

Solve these to find $m_0 = \frac{17}{2}$. 

(c) Solve $\pi = \pi P$ and $\sum \pi_i = 1$ to find $\pi = \frac{1}{19}(9, 8, 2)$.

2. **10 marks** Consider a Markov chain on $S = \{0, 1, 2, \ldots\}$ with transition probabilities

$$P_{n,n+1} = p \quad P_{n,0} = 1 - p$$

for some fixed $p$.

(a) Is this chain irreducible?
(b) Is this chain reversible?
(c) Is state 0 periodic or aperiodic? If periodic, what is the period?
(d) Suppose $X_0 = 0$, and let $T$ be the return time: the minimal $n > 0$ with $X_n = 0$. Find the probability mass function for $T$.

**Solution:**

(a) Yes. From 0 it is possible to get to $n$ in $n$ steps, and back in one step.
(b) No. For example $P_{02} = 0$ but $P_{20}! = 0$ so there is no $\pi$ which solved detailed balance.  
(c) Aperiodic since $P_{00} > 0$.  
(d) The chain returns at time $n$ if the first $n$ steps are exactly $0 \rightarrow 1 \rightarrow 2 \rightarrow \cdots \rightarrow (n - 1) \rightarrow n$. This has probability $P_{01}P_{02} \cdots P_{n-2,n-1}P_{n,0} = p^{n-1}(1 - p)$.

3. **10 marks** Consider a branching process where the number of children of an individual is $\xi = \text{Bin}(2, 1/3)$.

(a) Find the probability generating function for $\xi$.

(b) Find the probability that the process becomes extinct.

**Solution:**

(a) $G(s) = \mathbb{E}s^X = (1/3)^2s^2 + 2(1/3)(2/3)s + (2/3)^2s^0 = (s/3 + 2/3)^2$.

(b) This is the smallest solution to $s = G(s)$. The solutions are 1 and 4, so the answer is 1. (Also clear since $\mathbb{E}\xi = 2/3 \leq 1$.

4. **10 marks** Consider a Markov chain with states $\{a, b, c, d, e, f, g\}$ and transition matrix

\[
\begin{pmatrix}
a & b & c & d & e & f & g \\
a & 0 & 0 & 1 & 0 & 0 & 0 \\
b & 1/4 & 1/4 & 1/4 & 1/4 & 0 & 0 \\
c & 1 & 0 & 0 & 0 & 0 & 0 \\
d & 0 & 0 & 0 & 1/2 & 1/2 & 0 \\
e & 0 & 0 & 0 & 0 & 1/2 & 1/2 \\
f & 0 & 0 & 0 & 1/2 & 0 & 1/2 \\
g & 0 & 1/3 & 0 & 0 & 0 & 0 & 1/3
\end{pmatrix}
\]

(a) Draw the transition diagram, including the probability for each transition.

(b) Find the communicating classes.

(c) For each class, determine whether it is recurrent or transient, and whether it is periodic.

**Solution:** See figure. Classes are

- $\{a, c\}$: recurrent, with period 2.
- $\{b\}$: transient aperiodic.
- $\{d, e, f\}$: recurrent aperiodic.
- $\{g\}$: transient aperiodic.