MATH 303 – Stochastic Processes – midterm 1

February 2022

1. $\boxed{\text{10 marks}}$ Let X_n be a Markov chain on $S = \{0, 1, 2, 3\}$ with transition matrix

$$P = \begin{pmatrix} \frac{1}{2} & \frac{1}{2} & 0 & 0 \\ 0 & \frac{2}{3} & \frac{1}{3} & 0 \\ 0 & 0 & \frac{3}{4} & \frac{1}{4} \\ 1 & 0 & 0 & 0 \end{pmatrix}.$$

   (a) Find the 2-step transition matrix of X_n
   (b) What is the stationary distribution?
   (c) If $X_0 = 0$ what is the expected number of steps before $X_n = 3$?

Solution:

(a) This is $P^2 = \begin{pmatrix} \frac{1}{2} & \frac{7}{12} & \frac{1}{6} & 0 & \\ 0 & \frac{4}{9} & \frac{17}{36} & \frac{1}{12} & \\ 0 & 0 & \frac{9}{16} & \frac{3}{16} & 0 \\ \frac{1}{2} & \frac{1}{2} & 0 & 0 & \end{pmatrix}$

(b) Solve $\pi = \pi P$ and $\sum \pi_i = 1$ to find $\pi = \frac{1}{10}(2, 3, 4, 1)$.

(c) Let m_i be the mean from state i. Then

$$m_0 = 1 + \frac{1}{2}m_0 + \frac{1}{2}m_1 \qquad m_1 = 1 + \frac{2}{3}m_1 + \frac{1}{3}m_2 \qquad m_2 = 1 + \frac{3}{4}m_2 + \frac{1}{4}m_3.$$

Solve these to find $m_0 = 9$.

2. $\boxed{\text{10 marks}}$ Consider a Markov chain on $S = \{0, 1, 2, \dots\}$ with transition probabilities

$$P_{n,n+1} = p \qquad P_{n,0} = 1 - p$$

for some fixed p.

   (a) Is this chain irreducible?
   (b) Is this chain reversible?
   (c) Is state 0 periodic or aperiodic? If periodic, what is the period?
   (d) Suppose $X_0 = 0$, and let T be the return time: the minimal $n > 0$ with $X_n = 0$. Find the probability mass function for T.

Solution:

(a) Yes. From 0 it is possible to get to n in n steps, and back in one step.
(b) No. For example $P_{02} = 0$ but $P_{20}! = 0$ so there is no π which solved detailed balance.
(c) Aperiodic since $P_{00} > 0$.
(d) The chain returns at time n if the first n steps are exactly $0 \to 1 \to 2 \to \cdots \to (n-1) \to n$. This has probability $P_{01}P_{02} \dots P_{n-2,n-1}P_{n,0} = p^{n-1}(1-p)$.

3. $\boxed{\text{10 marks}}$ Consider a branching process where the offspring distribution for the number of chidren of an individual is $\mathbb{P}(\xi = n) = \frac{1}{3}\left(\frac{2}{3}\right)^n$ for $n = 0, 1, \dots$.

(a) Find the probability generating function for ξ.
(b) Find the probability that the process becomes extinct.

Solution:

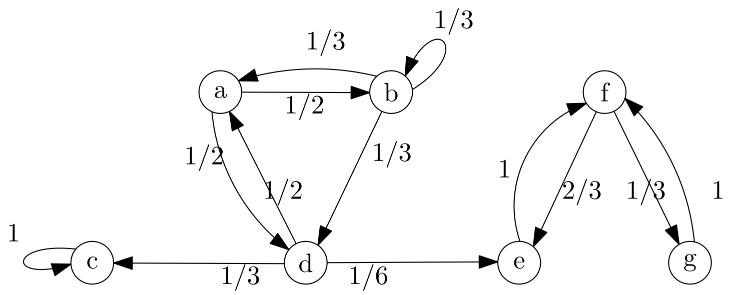
(a) $G(s) = \mathbb{E}s^X = \sum \frac{1}{3}\left(\frac{2}{3}\right)^n s^n = \frac{1}{3}\frac{1}{1-2s/3}$.
(b) This is the smallest solution to $s = G(s)$. The solutions are 1 and 1/2, so the answer is 1/2.

4. $\boxed{\text{10 marks}}$ Consider a Markov chain with states $\{a, b, c, d, e, f, g\}$ and transition matrix

$$\begin{array}{c} \\ a \\ b \\ c \\ d \\ e \\ f \\ g \end{array}
\begin{array}{c} \begin{array}{ccccccc} a & b & c & d & e & f & g \end{array} \\
\begin{pmatrix}
0 & \frac{1}{2} & 0 & \frac{1}{2} & 0 & 0 & 0 \\
\frac{1}{3} & \frac{1}{3} & 0 & \frac{1}{3} & 0 & 0 & 0 \\
0 & 0 & 1 & 0 & 0 & 0 & 0 \\
\frac{1}{2} & 0 & \frac{1}{3} & 0 & \frac{1}{6} & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 1 & 0 \\
0 & 0 & 0 & 0 & \frac{2}{3} & 0 & \frac{1}{3} \\
0 & 0 & 0 & 0 & 0 & 1 & 0
\end{pmatrix} \end{array}$$

(a) Draw the transition diagram, including the probability for each transition.
(b) Find the communicating classes.
(c) For each class, determine whether it is recurrent or transient, and whether it is periodic.

Solution: See figure. Classes are



- $\{a, b, d\}$: transient aperiodic.
- $\{c\}$: recurrent aperiodic.
- $\{e, f, g\}$: recurrent with period 2.