MATH 303 – Stochastic Processes – midterm 2

March 2022

1. $\boxed{\text{12 marks}}$ Let $N(t)$ be a Poisson process with rate λ. Find the following quantities. Briefly explain your reasoning.

   (a) $P(N(2) = 2 | N(5) = 10)$.

   (b) $P(N(10) = 6 | N(2) = 4)$.

   (c) $P(N(2) = N(6))$.

   (d) $P(N(2) = N(6) | N(4) = 3)$.

Solution:

(a) Each of 10 events in $(0, 5)$ is in $(0, 2)$ with probability $2/5$, so this is

$$P(Bin(10, 2/5) = 2) = \binom{10}{2}(2/5)^2(3/5)^8.$$

(b) 2 events in (2,10) have probability $e^{-8\lambda}\frac{(8\lambda)^2}{2!}$.

(c) No events in $(2, 6)$ has probability $e^{-4\lambda}$.

(d) This requires no events in $(2, 4)$ or in $(4, 6)$. Given $N(4) = 3$, the first has probability $(2/4)^3$ and the second has probability $e^{-2\lambda}$, so $P = (2/4)^3e^{-2\lambda}$.

2. $\boxed{\text{16 marks}}$ Buses arrive at a station according to a Poisson process with rate $\lambda = 0.3$ (with time measured in minutes). Each bus is a #17 with probability $1/3$ and #99 with probability $2/3$. Let $N_i(t)$ be the number of buses of type i up to time t.

   (a) What is the distribution of the number of #99 buses who pass in a 20 minute interval?

   (b) What is the joint distribution of $N_{17}(60)$ and $N_{99}(60)$?

   (c) If you arrive at the station and see a #99 bus leaving, how long on average do you need to wait for the next #99 (ignoring #17 buses)?

   (d) A bus at time t is full with probability $p(t) = t/60$ for $t \in [0, 60]$. What is the distribution of the number of such buses that pass during $[0, 60]$?

Solution:

(a) #99 buses are a Poisson process with rate $\lambda p_{99} = 0.2$. In 20 minutes this gives $\mathrm{Poi}(20 \cdot 0.2) = \mathrm{Poi}(4)$ buses.

(b) These are independent Poisson variables with means $60\lambda p_{17} = 6$ and $60\lambda p_{99} = 12$.

(c) The fact that a bus just left is irrelevant. The time for the next 99 is Exp(0.2), so the average is $1/0.2 = 5$ minutes.

(d) The number of full buses is Poisson with mean

$$\int_0^{60} \lambda p_{full}(s) ds = \int_0^{60} \frac{0.3t}{60} = \frac{0.3 \cdot 60^2}{60 \cdot 2} = 9.$$

3. $\boxed{\text{12 marks}}$ A Markov chain on state space $\{0, 1, 2 \dots\}$ has transition probabilities

$$P_{n,n+1} = \lambda_n \qquad P_{n,n-1} = \tfrac{1}{3} \qquad P_{n,n} = \tfrac{2}{3} - \lambda_n,$$

For some λ_n, except that $P_{0,-1} = 0$ and $P_{0,0} = 1 - \lambda_0$.

(a) What is λ_n if the stationary measure for this Markov chain is Poi(1)?

(b) Suppose $\lambda_n = \frac{1}{n+1}$. Write down equations that determine the stationary distribution π, and find the stationary distribution.

Solution: Using detailed balance we have $\pi_n \lambda_n = \pi_{n+1}/3$.

(a) In this case we have

$$\frac{\lambda_n e^{-1}}{n!} = \frac{e^{-1}}{3(n+1)!},$$

so $\lambda_n = \frac{1}{3(n+1)}$.

(b) Detailed balance gives $\pi_{n+1} = \frac{3}{n+1}\pi_n$. By multiplying this we get

$$\pi_n = \frac{3}{n}\frac{3}{n-1} \cdots \frac{3}{1}\pi_0 = \frac{3^n}{n!}\pi_0.$$

To sum to 1 we must have $\pi_0 = e^{-3}$, and this gives a Poi(3) stationary distribution.

4. $\boxed{\text{10 marks}}$ A continuous time Markov chain with states $1, 2, 3$ has $v_i = 2i$, and $P_{i,j} = 1/2$ for every $i \neq j$.

(a) If $X_0 = 1$, what is the probability that the chain makes no jumps before time 1?

(b) If $X_0 = 1$ and the first jump is at time T_1 , what is the probability of no second jump before time $T_1 + 1$?

Solution:

(a) The jump from 1 is after time $\mathrm{Exp}(v_1)$, so the probability of no jum by time 1 is $e^{-v_1} = e^{-2}$.

(b) The first jump is to 2 or 3 with probability 1/2 each. If the jump is to 2, then the probability of no second jump by T_1+1 is e^{-v_2}. If to 3, it is e^{-v_3}. Therefore

$$P = \tfrac{1}{2}e^{-v_2} + \tfrac{1}{2}e^{-v_3} = \frac{e^{-4}+e^{-6}}{2}.$$