MATH 303 – Stochastic Processes – midterm 2

March 2022

1. [12 marks] Let $N(t)$ be a Poisson process with rate λ. Find the following quantities. Briefly explain your reasoning.

   (a) $P(N(8) = 6|N(3) = 4)$.

   (b) $P(N(3) = 4|N(8) = 6)$.

   (c) $E(N(8) = N(10))$.

   (d) $P(N(2) = N(8)|N(6) = 2)$.

Solution:

(a) 2 events in (3,8) have probability $e^{-5\lambda}\frac{(5\lambda)^2}{2!}$.

(b) Each of 6 events in $(0, 8)$ is in $(0, 3)$ with probability 3/8, so this is

$$P(Bin(6, 3/8) = 4) = \binom{6}{4}(3/8)^4(5/8)^2.$$

(c) No events in $(8, 10)$ has probability $e^{-2\lambda}$.

(d) This requires no events in $(2, 6)$ or in $(6, 8)$. Given $N(6) = 2$, the first has probability $(2/6)^2$ and the second has probability $e^{-2\lambda}$, so $P = (2/6)^2e^{-2\lambda}$.

2. [16 marks] Customers arrive at a restaurant at rate $\lambda = 20$ per hour. Each customer stays for a random amount of time: either 0.5 hour (type 1) or one hour (type 2) with probability 1/2 each. The restaurant lets customers enter for 10 hours a day ($t = 0$ to $t = 10$). After that then no more customers can enter, but the restaurant remains open until all present customers leave. Let $N_i(t)$ be the number of customers of type i who enter by time $t \in [0, 10]$.

   (a) The cook takes 10 minute bathroom break. What is the probability that no customers enter during the brake?

   (b) The manager leaves for some errands for an hour. What is the probability that at least one customer comes **and** leaves while the manager is not there?

   (c) What is the distribution of the number of type 2 customers persent at $t = 10$?

   (d) What is the probability that the restaurant is still open at $t = 10.75$ (3/4 hour after $t = 10$)?

Solution:

(a) No enrance in $1/6$ of an hour has probability $e^{-20/6}$.

(b) Only a type 1 customer can arrive and leave within one hour, if they arrive in the first half hour. The type 1 customers are a Poisson process with rate 10, so the probability of this is $1 - e^{-10(1/2)}$.

(c) These are type 2 customers who arrive within the last hour, so it is Poi(10).

(d) The resturant is still open at 10.75 if a type 2 customer arrived after time 9.75. Therefore

$$P = P(\text{Poi}(10/4) > 0) = 1 - e^{-10/4}.$$

3. [12 marks] A Markov chain on state space $\{1, 2, 3 \ldots\}$ has transition probabilities

$$P_{n,n+1} = \lambda_n \qquad P_{n,n-1} = \tfrac{1}{2} \qquad P_{n,n} = \tfrac{1}{2} - \lambda_n,$$

For some λ_n, except that $P_{1,0} = 0$ and $P_{1,1} = 1 - \lambda_1$.

(a) What is λ_n if the stationary measure for this Markov chain is $\pi_n = \frac{1}{n(n+1)}$

(b) Suppose $\lambda_n = \frac{1}{3}$. Write down equations that determine the stationary distribution π, and find the stationary distribution.

Solution: Using detailed balance we have $\pi_n \lambda_n = \pi_{n+1}/2$.

(a) In this case we have

$$\frac{\lambda_n}{n(n+1)} = \frac{1}{2(n+1)(n+2)},$$

so $\lambda_n = \frac{n}{2(n+2)}$.

(b) Detailed balance gives $\pi_{n+1} = (2/3)\pi_n$, so $\pi_n = (2/3)^{n-1}\pi_1$. To add up to 1 we must have $\pi_1 = 1/3$. (This is Geom(1/3).)

4. [10 marks] A continuous time Markov chain with states 1, 2, 3 has $v_i = 2i$, and $P_{i,j} = 1/2$ for every $i \neq j$.

(a) If $X_0 = 1$, what is the probability that the chain makes no jumps before time 1?

(b) If $X_0 = 1$ and the first jump is at time T_1 , what is the probability of no second jump before time $T_1 + 1$?

Solution:

(a) The jump from 1 is after time $\text{Exp}(v_1)$, so the probability of no jum by time 1 is $e^{-v_1} = e^{-2}$.

(b) The first jump is to 2 or 3 with probability 1/2 each. If the jump is to 2, then the probability of no second jump by $T_1 + 1$ is e^{-v_2}. If to 3, it is e^{-v_3}. Therefore

$$P = \tfrac{1}{2}e^{-v_2} + \tfrac{1}{2}e^{-v_3} = \frac{e^{-4} + e^{-6}}{2}.$$