Stochastic Processes
Sample midterm 1

Note: These are sample questions, and is not an indication of the midterm length.

Problem 1. Let X_n be a Markov chain on $S = \{0, 1, 2\}$ with transition matrix

$$P = \frac{1}{2}\begin{pmatrix} 0 & 1 & 1 \\ 1 & 1 & 0 \\ 1 & 0 & 1 \end{pmatrix}.$$

(a) Find the 2-step transition matrix of X_n
(b) We assume for the rest of the problem that $X_0 = 0$. Find $\mathbb{E}(X_2)$.
(c) What is the mean number of steps until X_n visits 2?

Problem 2. We consider a polygon with 5 vertices, labeled clockwise from 1 to 5. Define a Markov chain $(X_n)_{n\geq 0}$ on the vertices as follows: at each turn n, one rolls an unbiased dice with 6 sides and moves clockwise a number of steps equal to the outcome of the dice. For example, if X_n is at 1 and one rolls 1,2,3,4,5, or 6, then X_{n+1} is 2, 3, 4, 5, 1, 2, respectively.
(a) Write the transition matrix of X.
(b) Is the chain ergodic? (justify!)
(c) What is the mean number of turns it takes to re-visit its starting state?

Problem 3. Recall that j is accessible from i (Denoted $i \to j$) if there exists $m > 0$ such that $P_{ij} > 0$. Show that if $i \to j$ and $j \to k$, then $i \to k$.

Problem 4. Give the communicating classes and classify the states (periodicity, positive or null-recurrence, transience) of the Markov chain associated with a given transition diagram. (briefly justify all claims.) (The sample midterm does not have a specific figure, so draw a few and make sure you can analyze them.)

Problem 5. Consider the Markov chain on $\mathbb{N}$ with transition probabilities $P_{n,n+1} = \frac{1}{n+1}$ and $P_{n0} = \frac{n}{n+1}$.
(a) If π is a stationary distribution, show that $\pi_0 = \pi_n \cdot n!$ for all n.
(b) Is this transient, positive recurrent or null recurrent? If positive recurrent, find the stationary distribution. (Recall $\sum_{n=0}^{\infty} \frac{1}{n!} = e$.)
(c) Let T be the first time to re-visit 0, starting at 0. For $k \geq 1$, find $P(T = k)$ and verify that $\mathbb{E}(T) = 1/\pi_0$. (Remark: drawing the transition diagram can help.)

Problem 6. Let X_n be a Markov chain on $S = \{0, 1, 2\}$ with transition matrix

$$P = \begin{pmatrix} 1/3 & 1/3 & 1/3 \\ 1/2 & 0 & 1/2 \\ 1/3 & 2/3 & 0 \end{pmatrix}.$$

Is this reversible? (Justify!)

Problem 7. Consider a branching process with offspring distribution ξ with $P(\xi = 0) = p$ and $P(\xi = 2) = q$ for some p, q with $p + q = 1$.
(a) What is $\mathbb{E}(Z_n)$?
(b) What is the probability that $Z_3 = 0$?
(c) What is the probability of eventual extinction?