Stochastic Processes
Sample midterm 2 hints

Note: These are sample questions, and is not an indication of the midterm length.

Problem 1. Let $\{N(t)\}_{t\geq 0}$ be a rate λ Poisson process, and $s < t$ postitive numbers.
(a) Find $P(N(t) > N(s))$
(b) Find $P(N(s) = 0, N(t) = 3)$.
(c) Find $E[N(t)|N(s) = 4]$.
(d) Find $E[N(s)|N(t) = 4]$.
(e) Find the joint probability mass function of $N(s), N(t)$.
(f) What is $P(N(1) = 1, N(2) = 2, N(3) = 3, N(4) = 4)$?
(g) Let S_i be the time of the ith event in N. What is $E(S_n)$?
(h) For a given t, what is $E(S_1|N(t) = 2)$
(i) For a given t, what is $E(S_4|N(t) = 2)$
(j) Suppose that each event is type 1 with probability p, and type 2 with probability $1 - p$. Let $N_i(t)$ be the number of type i events by time t. What is $E[N(t)|N_1(t + s) = k]$ for some integer k?

Solution. Note: many of these have several methods.
(a) This requires no events in (s, t), so $P = e^{-\lambda(t-s)}$.
(b) This requires no events in $(0, s)$ and 3 in (s, t), so $P = e^{-\lambda s}e^{-\lambda(t-s)}\frac{(\lambda(t-s))^3}{3!}$.
(c) $N(t) = N(s) + \mathrm{Poi}(\lambda(t - s))$, so the expectation is $4 + \lambda(t - s)$.
(d) Each of the 4 events is uniform in $[0, t]$, so has probability s/t of being before time s. Therefore the expectation is $4s/t$.
(e) $N(s)$ and $N(t - s)$ are independents, so

$$p(k, l) = e^{-\lambda s}\frac{(\lambda s)^k}{k!}e^{-\lambda(t-s)}\frac{(\lambda(t-s))^(k-l)}{(k-l)!}.$$

(f) This requires one event in each of $(0, 1), (1, 2), (2, 3), (3, 4)$, so the probability is $(\lambda e^{-\lambda})^4$.
(g) S_n is a sum of n independent $\mathrm{Exp}(\lambda)$ variables, so its expectation is n/λ.
(h) Given that there are two events up to time t, they are uniform in $[0, t]$, so the first is the minimum of two variables uniform in $[0, t]$. This has density $2x/t^2$ on $[0, t]$, and expectation $t/3$.
(i) If there are two events up to time t, then the time to get 2 events after time t has mean $2/\lambda$, and so the expectation is $t + 2/\lambda$.
(j) There are k events of type 1 by time $t + s$. Each of these is in $[0, t]$ with probability $t/(t + s)$ independently, so the conditional expected number of type 1 events in $[0, t]$ is $\frac{tk}{t+s}$. The expected number of type 2 events by time t is $(1 - p)\lambda t$, and together we get $\frac{tk}{t+s} + (1 - p)\lambda t$.

Problem 2. Starting from time 0, the #99 bus arrives according to a rate λ Poisson process. Passengers arrive at the station according to an independent rate μ Poisson process. When a bus arrives, all waiting passengers instantly enter the bus.
(a) Find the p.m.f. for the number of passengers entering a bus.
(b) Find the p.m.f. for the number of passengers entering the a bus given that the time between it and the previous bus is t.
(c) Suppose that the buses have been going for ever. You arrive at the bus stop at 9:00. What is the distribution of the time since the last bus had gone?
(d) You arrive at the bus stop at 9:00. What is the expected number of passengers waiting at the station?

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Solution.

(a) At any time, the probability that the next event is a passenger arriving is $\frac{\mu}{\mu+\lambda}$ and the probability that a bus arrives is $\frac{\lambda}{\mu+\lambda}$ (why?). Suppose a bus just left. The number of events before the next bus comes is therefore geometric, and so

$$\mathbb{P}(n \text{ passengers}) = \left(\frac{\mu}{\mu+\lambda}\right)^n \frac{\lambda}{\mu+\lambda}.$$

(b) If the time since the last bus is known to be t, the number of passengers is $\text{Poi}(\mu t)$.

(c) For a fixed t, the probability that there was no bus in time (s,t) is $e^{-\lambda(t-s)}$, so the probability the last bus was before time $9-s$ is $e^{-\lambda s}$: It is $\text{Exp}(\lambda)$.

(d) Given the time since the last bus is S, the expectation is μS. Since the time since the last bus is $\text{Exp}(\lambda)$, we get

$$E(X) = E(E(X|S)) == E(\mu S) = \mu/\lambda.$$

Problem 3. A small barbershop has room for at most two customers (including the one the barber is working on). Potential customers arrive at a Poisson rate of 3 per hour, and the successive service times are independent exponential random variables with mean 4 per hour. If a customer arrives and there is no room in the shop, they leave.

Set this up as a birth and death process: Find the rates for jumping out of each state and the jump probabilities.

Solution. The state space is $\{0,1,2\}$. From 0, customers arrive at rate 3, so $\lambda_0 = 3$. From 1, customers arrive at rate 3, so $\lambda_1 = 3$. The rate of service is 4, so $\mu_1 = 4$. From 2, no arrivals are possible, so $\lambda_2 = 0$, and $\mu_2 = 4$.

This gives $v_0 = 3$, $v_1 = 7$, and $v_2 = 4$. The jump probabilities are $P_{01} = P_{21} = 1$, and $P_{10} = \frac{4}{7}$, $P_{12} = \frac{3}{7}$ and all others are 0.

Mode detail: For example if there is one customer at the shop, the time for another arrival is Exp(3) and the time for the customer to eave is Exp(4). The first of these happens at time $\text{Exp}(3+4)$, so $v_1 = 7$. The probability that the first event is a customer arriving is $\frac{3}{3+4}$, so that's P_{12}.