Stochastic Processes
Sample midterm 2 hints

Note: These are sample questions, and is not an indication of the midterm length.

Problem 1. Let \( \{N(t)\}_{t \geq 0} \) be a rate \( \lambda \) Poisson process, and \( s < t \) positive numbers.
(a) Find \( P(N(t) > N(s)) \)
(b) Find \( P(N(s) = 0, N(t) = 3) \).
(c) Find \( E[N(t)|N(s) = 4] \).
(d) Find \( E[N(s)|N(t) = 4] \).
(e) Find the joint probability mass function of \( N(s), N(t) \).
(f) What is \( P(N(1) = 1, N(2) = 2, N(3) = 3, N(4) = 4) \)?
(g) Let \( S_i \) be the time of the \( i \)th event in \( N \). What is \( E(S_i) \)?
(h) For a given \( t \), what is \( E(S_{1}|N(t) = 2) \)
(i) For a given \( t \), what is \( E(S_{4}|N(t) = 2) \)
(j) Suppose that each event is type 1 with probability \( p \), and type 2 with probability \( 1 - p \). Let \( N_i(t) \) be the number of type \( i \) events by time \( t \). What is \( E[N(t)|N_i(t + s) = k] \) for some integer \( k \)?

Solution. Note: many of these have several methods.
(a) This requires no events in \( (s,t) \), so \( P = e^{-\lambda(t-s)} \).
(b) This requires no events in \( (0,s) \) and 3 in \( (s,t) \), so \( P = e^{-\lambda_s}e^{-\lambda(t-s)} \frac{(\lambda (t-s))^3}{3!} \).
(c) \( N(t) = N(s) + \text{Poi}(\lambda(t-s)) \), so the expectation is \( 4 + \lambda(t-s) \).
(d) Each of the 4 events is uniform in \( [0,t] \), so has probability \( s/t \) of being before time \( s \). Therefore the expectation is \( 4s/t \).
(e) \( N(s) \) and \( N(t-s) \) are independents, so
\[
p(k,l) = e^{-\lambda_s} \left( \frac{(\lambda s)^k}{k!} \right) e^{-\lambda(t-s)} \left( \frac{(\lambda (t-s))^{l}}{(k-l)!} \right).
\]
(f) This requires one event in each of \( (0,1), (1,2), (2,3), (3,4) \), so the probability is \( (\lambda e^{-\lambda})^4 \).
(g) \( S_n \) is a sum of \( n \) independent \( \text{Exp}(\lambda) \) variables, so its expectation is \( n/\lambda \).
(h) Given that there are two events up to time \( t \), they are uniform in \( [0,t] \), so the first is the minimum of two variables uniform in \( [0,t] \). This has density \( 2x/t^2 \) on \( [0,t] \), and expectation \( t/3 \).
(i) If there are two events up to time \( t \), then the time to get 2 events after time \( t \) has mean \( 2/\lambda \), and so the expectation is \( t + 2/\lambda \).
(j) There are \( k \) events of type 1 by time \( t+s \). Each of these is in \( [0,t] \) with probability \( t/(t+s) \) independently, so the conditional expected number of type 1 events in \( [0,t] \) is \( \frac{tk}{t+s} \). The expected number of type 2 events by time \( t \) is \( (1-p)\lambda t \), and together we get \( \frac{tk}{t+s} + (1-p)\lambda t \).

Problem 2. Starting from time 0, the #99 bus arrives according to a rate \( \lambda \) Poisson process. Passengers arrive at the station according to an independent rate \( \mu \) Poisson process. When a bus arrives, all waiting passengers instantly enter the bus.
(a) Find the p.m.f. for the number of passengers entering a bus.
(b) Find the p.m.f. for the number of passengers entering the a bus given that the time between it and the previous bus is \( t \).
(c) Suppose that the buses have been going for ever. You arrive at the bus stop at 9:00. What is the distribution of the time since the last bus had gone?
(d) You arrive at the bus stop at 9:00. What is the expected number of passengers waiting at the station?
Solution.

(a) At any time, the probability that the next event is a passenger arriving is $\frac{\mu}{\mu + \lambda}$ and the probability that a bus arrives is $\frac{\lambda}{\mu + \lambda}$ (why?). Suppose a bus just left. The number of events before the next bus comes is therefore geometric, and so

$$P(n \text{ passengers}) = \left( \frac{\mu}{\mu + \lambda} \right)^n \frac{\lambda}{\mu + \lambda}.$$ 

(b) If the time since the last bus is known to be $t$, the number of passengers is $\text{Poi}(\mu t)$.

(c) For a fixed $t$, the probability that there was no bus in time $(s, t)$ is $e^{-\lambda(t-s)}$, so the probability the last bus was before time $9 - s$ is $e^{-\lambda s}$: It is $\text{Exp}(\lambda)$.

(d) Given the time since the last bus is $S$, the expectation is $\mu S$. Since the time since the last bus is $\text{Exp}(\lambda)$, we get

$$E(X) = E(E(X|S)) = E(\mu S) = \mu/\lambda.$$ 

**Problem 3.** A small barbershop has room for at most two customers (including the one the barber is working on). Potential customers arrive at a Poisson rate of 3 per hour, and the successive service times are independent exponential random variables with mean 4 per hour. If a customer arrives and there is no room in the shop, they leave.

Set this up as a birth and death process: Find the rates for jumping out of each state and the jump probabilities.

**Solution.** The state space is $\{0, 1, 2\}$. From 0, customers arrive at rate 3, so $\lambda_0 = 3$. From 1, customers arrive at rate 3, so $\lambda_1 = 3$. The rate of service is 4, so $\mu_1 = 4$. From 2, no arrivals are possible, so $\lambda_2 = 0$, and $\mu_2 = 4$.

This gives $v_0 = 3$, $v_1 = 7$, and $v_2 = 4$. The jump probabilities are $P_{01} = P_{21} = 1$, and $P_{10} = \frac{4}{7}$, $P_{12} = \frac{3}{7}$ and all others are 0.

Mode detail: For example if there is one customer at the shop, the time for another arrival is $\text{Exp}(3)$ and the time for the customer to eave is $\text{Exp}(4)$. The first of these happens at time $\text{Exp}(3 + 4)$, so $v_1 = 7$. The probability that the first event is a customer arriving is $\frac{3}{3+4}$, so that’s $P_{12}$. 
