Mathematical Game theory

Assignment 1, due 2022-01-21

Problem 1. A Markov chain with state space $S = \{0, 1, 2\}$ has the transition matrix

$$P = \begin{pmatrix} .2 & .3 & .5 \\ 0 & .4 & .6 \\ .7 & .3 & 0 \end{pmatrix}.$$

If X_0 has distribution vector $(.6, .4, 0)$, find the distribution vectors of X_1, X_2.

Problem 2. There are 6 coins on a table, each showing heads (H) or tails (T). In each step we

- Select uniformly one of the coins.
- If it is heads, toss it and replace on the table (with random side).
- If it is tails, toss it. If it comes up heads leave it at that. If if comes up tails, toss it a second time, and leave the result as it is.

Let X_n be the number of heads showing after n such steps.

(a) Determine the transition probabilities for this Markov chain.
(b) Draw the transition diagram and write the transition matrix.
(c) What is $\mathbb{P}(X_2 = 4 | X_0 = 5)$?

Problem 3. Jaine is a mechanic. Let X_n be the number of cars they need to fix. At each step, with probability 1/2 they manage to fix a car, and X decreases by 1. With probability 1/3 a new car is delivered for service and X increases by 1, and with probability 1/6 nothing happens and X stays the same. If X becomes 0, Jaine goes for lunch. If X becomes 3, the boss gets angry and fires Jaine.

(a) Suppose $X_0 = 1$. What is the probability that Jaine goes to lunch before they are fired?
(b) What would be the answer if the boss only fires Jaine when $X = 4$?

Problem 4. Consider the simple random walk on a triangle with vertices $\{0, 1, 2\}$. This is the Markov chain that at each step moves to one of the two states other than X_n. Prove that the n-step transition matrix is

$$P^n = \begin{pmatrix} a_n & b_n & b_n \\ b_n & a_n & b_n \\ b_n & b_n & a_n \end{pmatrix},$$

where $a_n = \frac{1+2(-1/2)^n}{3}$ and $b_n = \frac{1-(-1/2)^n}{3}$.

Problem 5. Alfonso, Beatrice, and Coraline play poker. They start with \$1, \$2, \$3 respectively. In each round, one of the three (chosen randomly) does not participate. The other two each bet \$1, and one of the two wins the bet (and the other loses), again randomly and equally likely. If a player has no money, they are eliminated and do not participate in future rounds. Eventually one player has all \$6, and the tournament ends.

(a) What is the probability that Alfonso is the eventual winner? (Hint: compare Alfonso's money to the gambler's ruin problem.)
(b) Let q_A be the probability that Alfonso is the first one to be eliminated, and similarly q_B, q_C for Beatrice and Coraline. Compute q_A, q_B, q_C. (Hint: Consider also what happens if the players have \$1,\$1,\$4.)

Extra practice problems Do not hand these in. (Feel free to ask for hints is stuck.)
Ross, chapter 4: problems 1,3,4,6,8,9.