Stochastic Processes
Assignment 1 solutions

Problem 1. A Markov chain with state space $S = \{0, 1, 2\}$ has the transition matrix

$$P = \begin{pmatrix} .2 & .3 & .5 \\ 0 & .4 & .6 \\ .7 & .3 & 0 \end{pmatrix}.$$

If X_0 has distribution vector $(.6, .4, 0)$, find the distribution vectors of X_1, X_2.

Solution. Let P be the transition matrix and v the given distribution of X_0. Then X_i has distribution $vP = (.12, .34, .54)$ and X_2 has distribution $vP^2 = (.402, .334, .264)$.

Problem 2. There are 6 coins on a table, each showing heads (H) or tails (T). In each step we

- Select uniformly one of the coins.
- If it is heads, toss it and replace on the table (with random side).
- If it is tails, toss it. If it comes up heads leave it at that. If if comes up tails, toss it a second time, and leave the result as it is.

Let X_n be the number of heads showing after n such steps.

(a) Determine the transition probabilities for this Markov chain.
(b) Draw the transition diagram and write the transition matrix.
(c) What is $\mathbb{P}(X_2 = 4 | X_0 = 5)$?

Solution.

(a) X decreases by one if we pick a coin showing heads, toss it and it comes tails. This has probability $\frac{X}{6} \cdot \frac{1}{2}$. X increases by one if we pick a coin showing tails, toss it and it comes heads either the first or second try. This has probability $\frac{6-X}{6} \cdot \frac{3}{4}$. Otherwise X stays the same. This gives

$$P_{i,i-1} = \frac{i}{12} \qquad p_{i,i+1} = \frac{3(6-i)}{24} \qquad P_{i,i} = \frac{6+i}{24}.$$

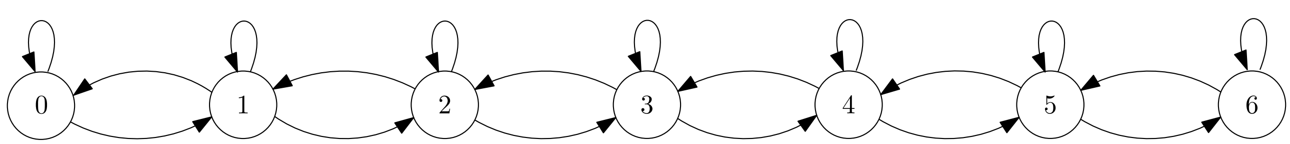
(b) See figure.

(c)

$$P = \frac{1}{24}\begin{pmatrix} 6 & 18 & 0 & 0 & 0 & 0 & 0 \\ 2 & 7 & 15 & 0 & 0 & 0 & 0 \\ 0 & 4 & 8 & 12 & 0 & 0 & 0 \\ 0 & 0 & 6 & 9 & 9 & 0 & 0 \\ 0 & 0 & 0 & 8 & 10 & 6 & 0 \\ 0 & 0 & 0 & 0 & 10 & 11 & 3 \\ 0 & 0 & 0 & 0 & 0 & 12 & 12 \end{pmatrix}$$

(d) This is

$$P^2_{54} = P_54P_44 + P_55P_54 = \frac{10}{24}\frac{10}{24} + \frac{11}{24}\frac{10}{24}.$$



Problem 3. Jaine is a mechanic. Let X_n be the number of cars they need to fix. At each step, with probability $1/2$ they manage to fix a car, and X decreases by 1. With probability $1/3$ a new car is delivered for service and X increases by 1, and with probability $1/6$ nothing happens and X stays the same. If X becomes 0, Jaine goes for lunch. If X becomes 3, the boss gets angry and fires Jaine.

(a) Suppose $X_0 = 1$. What is the probability that Jaine goes to lunch before they are fired?

(b) What would be the answer if the boss only fires Jaine when $X = 4$?

Solution.

(a) Let q_i be the probability of going to lunch without getting fired if there are i cars to fix. Then $q_0 = 1$ and $q_3 = 0$. For the others we have

$$q_1 = \frac{1}{2}q_0 + \frac{1}{3}q_2 + \frac{1}{6}q_1$$
$$q_2 = \frac{1}{2}q_1 + \frac{1}{3}q_3 + \frac{1}{6}q_2$$

The solution to these is $q_1 = \frac{15}{19}, q_2 = \frac{9}{19}$.

(b) Replace $q_3 = 0$ by $q_4 = 0$ and $q_3 = \frac{1}{2}q_2 + \frac{1}{3}q_4 + \frac{1}{6}q_3$. The solution now is $q_1 = 57/65$, $q_2 = 45/65$, $q_3 = 27/65$.

Note. This is exactly the same as the gambler's ruin problem, except that some steps nothing happens. However, such steps do not make a difference to the eventual outcome, so we can consider only the times there is a change. With probability $3/5$ X decreases, and with probability $2/5$ increases at those times.

Problem 4. Consider the simple random walk on a triangle with vertices $\{0, 1, 2\}$. This is the Markov chain that at each step moves to one of the two states other than X_n. Prove that the n-step transition matrix is

$$P^n = \begin{pmatrix} a_n & b_n & b_n \\ b_n & a_n & b_n \\ b_n & b_n & a_n \end{pmatrix},$$

where $a_n = \frac{1+2(-1/2)^n}{3}$ and $b_n = \frac{1-(-1/2)^n}{3}$.

Solution. We prove this by induction. For $n = 1$ we have $a_1 = 0$ and $b_1 = 1/2$, and indeed this is the transition matrix of the Markov chain. If known for n, we need to show that

$$P \cdot \begin{pmatrix} a_n & b_n & b_n \\ b_n & a_n & b_n \\ b_n & b_n & a_n \end{pmatrix} = \begin{pmatrix} a_{n+1} & b_{n+1} & b_{n+1} \\ b_{n+1} & a_{n+1} & b_{n+1} \\ b_{n+1} & b_{n+1} & a_{n+1} \end{pmatrix},$$

This requires showing that $a_{n+1} = \frac{1}{2}b_n + \frac{1}{2}b_n$ and that $b_{n+1} = \frac{1}{2}a_n + \frac{1}{2}a_n$. Both of these are easy to verify for the given a_n, b_n. (Plug in and check.)

Problem 5. Alfonso, Beatrice, and Coraline play poker. They start with \$1, \$2, \$3 respectively. In each round, one of the three (chosen randomly) does not participate. The other two each bet \$1, and one of the two wins the bet (and the other loses), again randomly and equally likely. If a player has no money, they are eliminated and do not participate in future rounds. Eventually one player has all \$6, and the tournament ends.

(a) What is the probability that Alfonso is the eventual winner? (Hint: compare Alfonso's money to the gambler's ruin problem.)

(b) Let q_A be the probability that Alfonso is the first one to be eliminated, and similarly q_B, q_C for Beatrice and Coraline. Compute q_A, q_B, q_C. (Hint: Consider also what happens if the players have \$1,\$1,\$4.)

Solution.

(a) Alfonso's money in each round where he participates increases or decreases by 1 with equal probability. The probability that Alfonso reaches 6 before reaching 0 is given by the gambler's ruin result to be 1/6.

(b) Let q_{abc} be the probability that Alfonso is the first eliminated if the amounts they have are a, b, c. If anyone has 0 then either it's Alfonso and the value is 1, or it is someone else and the result is 0. The unknowns are

$$q_{114}, q_{123}, q_{132}, q_{141}, q_{213}, q_{222}, q_{231}, q_{312}, q_{321}, q_{411}.$$

These can be arranged as in the figure below, and each is the average of the 6 neighbouring values. These can be simplified since $q_{114} = q_{141}$, $q_{123} = q_{132}$, $q_{213} = q_{231}$, $q_{312} = q_{321}$ (since exchanging Beatrice and Coraline makes no difference. Also, $q_{222} = 1/3$ by symmetry. Also, $q_{123} + q_{213} + q_{312} = 1$ since exactly one of the three players must be the first eliminated, and also $q_{411} + 2q_{114} = 1$.
Letting $x = q_{123}$, $y = q_{213}$ and $z = q_{114}$, the others are all equal or $1 - x - y$ or $1 - 2z$ or $1/3$. We get several equations:

$$z = \frac{1}{3} + \frac{1}{6}x + \frac{1}{6}y \qquad x = \frac{1}{3} + \frac{1}{6}y + \frac{1}{6}z + \frac{1}{6}x + \frac{1}{6}\frac{1}{3} \qquad y = \frac{1}{6}z + \frac{1}{6}x + \frac{1}{6}\frac{1}{3} + \frac{1}{6}(1 - x - y)$$

These have solution

$$q_{123} = x = \frac{121}{197} \qquad q_{212} = y = \frac{51}{197} \qquad q_{312} = 1 - x - y = \frac{25}{197} \qquad q_{114} = z = \frac{283}{591}.$$