Stochastic Processes
Assignment 2 solutions

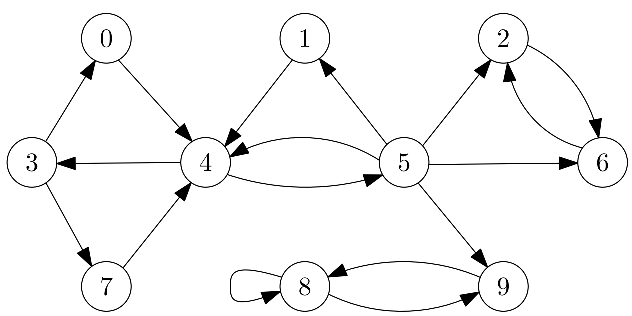
Problem 1. Consider the chains with state space the squares of a chess

(a) A chess king moves on it uniformly to an allowed position. Is this irreducible? periodic? recurrent?
(b) Same questions if the king is replaced by a bishop.
(c) Same questions if the king is replaced by a knight.

Solution.

(a) The king can reach any square from any other, so this is irreducible. It is possible to return to the starting square in 2 or 3 steps, so there is no period (aperiodic). This is recurrent (as is any irreducible Markov chain with finitely many states).
(b) This is reducible, since the bishop cannot move from a white square to a black square (or vice-versa). It is aperiodic and recurrent for the same reasons.
(c) This is irreducible: the knight can reach any square on the board. This one is periodic, with period 2, since the knight switches at every step from black to white and from white to black.

Problem 2. Consider the Markov chain with transition diagram below (all arrows have unspecified positive probability. What are the communicating classes? For each class determine if it is recurrent or transient and the period.



Solution.

- $\{0, 1, 3, 4, 5, 7\}$ is a transient, aperiodic class.
- $\{2, 6\}$ is a recurrent, periodic class (period 2).
- $\{8, 9\}$ is a recurrent, aperiodic class.

Problem 3. Let P be the transition matrix for an irreducible Markov chain. The lazy chain has the transition matrix $\hat{P} := (1 - a)I + aP$ for some constant $a \in (0, 1)$. Show that the following hold.

(a) $\hat{P}$ is a valid transition matrix, i.e., it is a stochastic matrix.
(b) If a Markov chain is irreducible and has some state i with $P_{i,i} > 0$, show that the chain is aperiodic. (In particular this applies for $\hat{P}$.)
(c) $\hat{P}$ has the same stationary distributions as P.

Solution.

(a) Each row of aP sums to a. Each row of $(1-a)I$ has $1-a$ on the diagonal and 0's elsewhere, so together rows sum to 1. Also, each entry is at least 0, so $\hat{P}$ is a stochastic matrix.

(b) Such a Markov chain is aperiodic since $P_{ii}^1 > 0$ so the lengths of cycles at i hav no common divisor. Other states are aperiodic since they communicate with i. Since $\hat{P}_{ii} = 1 - a + aP^{ii}$, this is always positive.

(c) If $\pi P = \pi$ then

$$\begin{aligned}\pi\hat{P} = \pi((1-a)I + aP) &= \pi(1-a)I + \pi a P \\ &= (1-a)\pi + a\pi = \pi.\end{aligned}$$

Problem 4. Consider a Markov chain with n states arranged in a circle. At each step the chain jumps one step clockwise with probability 2/3 and one step anticlockwise with probability 1/3.

(a) Show that this is periodic if n is even and aperiodic if n is odd.

(b) What is the stationary measure for this chain?

Solution.

(a) If n is even, color the states along the cycle alternating black and white. The chain switches color at each step, so can only return to its starting point afteran even number of steps. Thus the period is 2. It is possible to return after 2 steps by going orward and backward (for all n). It is also possible to return after n steps by going around the circle once. The greatest common divisor of a set that includes 2 and and odd n is 1.

(b) It is easy to see that $\pi_i = \frac{2}{3}\pi_{i-1} + \frac{1}{3}\pi_{i+1}$ is satisfied if the π_i are all equal. To add up to 1 they are each $1/n$. Thus the stationary distribution is $\pi_i = 1/n$ for all i.

Problem 5. Recall the chain from last week: There are 6 coins on a table, each showing heads (H) or tails (T). In each step we

- Select uniformly one of the coins.
- If it is heads, toss it and replace on the table (with random side).
- If it is tails, toss it. If it comes up heads leave it at that. If if comes up tails, toss it a second time, and leave the result as it is.

Let X_n be the number of heads showing after n such steps.
Show that the stationary distribution is $\mathrm{Bin}(6, q)$ for some q.

Solution. We can check directly from the transition probabilities from last week that $\pi = \pi P$, if $\pi_i = \binom{6}{i}(3/5)^i(2/5)^{6-i}$.

More simply, observe that each coin on its own when selected is a Markov chain with states $\{H, T\}$ and transitions $P_{HT} = 1/2$ and $P_{TH} = 3/4$. For the two state Markov chain we saw in lecture that

$$\pi_H = \frac{P_{TH}}{P_{TH} + P_{HT}} = \frac{3}{5} \qquad \pi_T = \frac{P_{HT}}{P_{TH} + P_{HT}} = \frac{2}{5}.$$

Therefore at stationarity each coin is H with probability 3/5, and coins are independent. The number of heads is therefore binomial.