Stochastic Processes

Assignment 3, due 2022-02-04

Note: From this assignment onward, start each problem on a **new page**. (Multiple pages for a single problem are fine if needed.)

Problem 1. Let C be a communicating class of a Markov chain. We say that C is closed if $P_{ij} = 0$ for all states $i \in C$ and $j \notin C$. In other words, a communicating class is closed if there is no escape from that class.

(a) Show that a finite communicating class C is closed if and only if its states are recurrent.
(b) Find an example of a Markov chain with no closed communicating class.
(c) Show that a finite recurrent communicating class C is positive-recurrent.

Problem 2. Yin and Yang play table tennis. For each point one of the players serves. The winner of a point becomes the server of the next point. Suppose that Yin wins each point she serves with probability p, and wins each point that Yang serves with probability q.

(a) Find the proportion of points that are won by Yin
(b) Find the proportion of time that the server swaps.

Problem 3. A chess knight starts at the bottom left of a chess board and performs random moves. At each stage, she picks one of the available legal moves with equal probability, independently of the earlier moves. Let X_n be her position after n moves. What is the mean number of moves before she returns to her starting square?

Problem 4. Kaine is building a house of cards. When there are i cards, a_i is the probability she adds another card without the house collapinsg, and otherwise it collapses and she starts from 0. Let X_n be the number of cards in the constructed house at time n. This is a Markov chain with state space $\mathbb{N} = 0, 1, 2, \dots$ with transition probabilities

$$P_{i,i+1} = a_i \qquad \text{and} \qquad p_{i,0} = 1 - a_i,$$

where a_i are numbers between 0 and 1.

Let $b_0 = 1$ and the b_i the product $b_i = a_0 a_1 \dots a_i$. Show that the chain is

(a) Recurrent if and only if $\lim_{i\to\infty} b_i = 0$;
(b) Positive recurrent if and only if $\sum_{i=0}^{\infty} b_i < infty$;
(c) Find the stationary distribution in the positive recurrent case.

Problem 5. Maine (the person, not the state) has 4 umbrellas, some at the office and some at home. Every time she commutes, if it rains and there is an umbrella by her, she takes one. If all the umbrellas are at the other location, she gets wet. Suppose each morning and afternoon it rains with some probability q, independently of all other times. Let X_n be the number of umbrellas at Maine's current location after n trips. For example, if $X_n = 3$ there is one umbrella at the other location. If it is raining, she takes one and $X_{n+1} = 2$ otherwise $X_{n+1} = 1$.

(a) Find the stationary distribution for X_n.
(b) Find the value of q for which the probability of getting wet on an average trip is maximized.

Extra practice problems Do not hand these in. (Feel free to ask for hints is stuck.)

Ross, chapter 4: problems 32,36,41,49,26,27.