Stochastic Processes
Assignment 3 solutions

Problem 1. Let C be a communicating class of a Markov chain. We say that C is closed if $P_{ij} = 0$ for all states $i \in C$ and $j \notin C$. In other words, a communicating class is closed if there is no escape from that class.

(a) Show that a finite communicating class C is closed if and only if its states are recurrent.
(b) Find an example of a Markov chain with no closed communicating class.
(c) Show that a finite recurrent communicating class C is positive-recurrent.

Solution.

(a) If a class C is closed, then the chain never leaves it, so spends infinite time in C. If C is finite, it must spend infinite time in some state in C, so that state is recurrent. By a theorem, all other states in C are also recurrent.
Conversely, if C is not closed, then at some state there is a probability of eiting C and never returning, so that state is transient, and so all of C is transient.
(b) Many examples exist. For example take $\mathbb{Z}$ with $P_{n,n+1} = 1$, so we always jump right. Each state is its own communicating class, and is not closed.
(c) In a finite recurrent communicating class with M states, the fraction of times at i must sum to 1, so at least one of them is at least $1/M$, and cannot all converge to 0, so they are not null-recurrent.

Problem 2. Yin and Yang play table tennis. For each point one of the players serves. The winner of a point becomes the server of the next point. Suppose that Yin wins each point she serves with probability p, and wins each point that Yang serves with probability q.

(a) Find the proportion of points that are won by Yin
(b) Find the proportion of time that the server swaps.

Solution. Let $X_0 = 0$ if Yin is serving for point n, and 1 if Yang is serving. This is a Markov chain with transitions $P_{00} = p$ and $P_{10} = q$ (and the others are $1 - p$ and $1 - q$.

(a) The stationary distribution is $\pi_0 = \frac{q}{q+(1-p)}$ and $\pi_1 = \frac{1-p}{q+(1-p)}$. Thus the fraction of time Yin is serving is π_0, which corresponds to points she won. Alternatively, she wins a point with probability $\pi_0 P_{00} + \pi_1 P_{10}$, which is the same (by stationarity).
(b) Serving swaps if Yin is serving and Yang wins a point or vice versa. This has probability

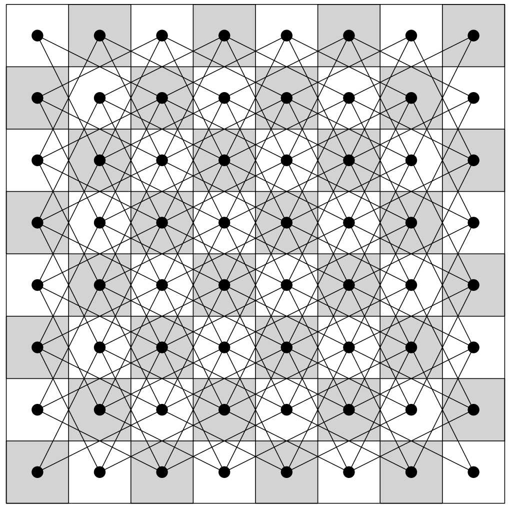
$$\pi_0 P_{01} + \pi_1 P_{10} = \frac{q}{q+1-p}(1-p) + \frac{1-p}{q+1-p}q = \frac{2q(1-p)}{q+1-p}.$$

Problem 3. A chess knight starts at the bottom left of a chess board and performs random moves. At each stage, she picks one of the available legal moves with equal probability, independently of the earlier moves. Let X_n be her position after n moves. What is the mean number of moves before she returns to her starting square?

Solution. This is a random walk on a graph as shown below. At each step the knight across a uniform edge from the current location. The stationarity distribution is $\pi_x = \frac{deg(x)}{2E}$ if $deg(x)$ is the degree of x and $E = 168$ is the total number of edges (42 in each of 4 directions). For the corner this gives $\pi_{a1} = \frac{2}{2\cdot 168} = \frac{1}{168}$. The average return time is $1/\pi_{a1} = 168$.

Problem 4. Kaine is building a house of cards. When there are i cards, a_i is the probability she adds another card without the house collapinsg, and otherwise it collapses and she starts from 0. Let X_n be the number of cards in the constructed house at time n. This is a Markov chain with state space $\mathbb{N} = 0, 1, 2, \ldots$ with transition probabilities

$$P_{i,i+1} = a_i \qquad \text{and} \qquad p_{i,0} = 1 - a_i,$$



where a_i are numbers between 0 and 1.

Let $b_0 = 1$ and the b_i the product $b_i = a_0 a_1 \dots a_i$. Show that the chain is

(a) Recurrent if and only if $\lim_{i\to\infty} b_i = 0$;

(b) Positive recurrent if and only if $\sum_{i=0}^{\infty} b_i < \infty$;

(c) Find the stationary distribution in the positive recurrent case.

Solution.

(a) The probability that the process moves from 0 to n without returning to 0 is $a_0 a_1 \dots a_{n-1} = b_{n-1}$. If $b_n \to 0$ then the chain must return to 0 at some time. Otherwise there is probability $\lim b_n$ that the chain never returns and so it is transient.

(b) The return time to 0 is at least n if and only if the chain reaches $n-1$ before returning to 0. This has probability b_{n-2} for $n \geq 2$. (It is 1 for $n = 1$.) We can use the result that for a positive integer random variable T we have $\mathbb{E}(T) = \sum_{n=1}^{\infty} P(T \geq n)$. In our case we get that $\mathbb{E}(T) = 1 + \sum b_n$, so this is finite if and only if $\sum b_n < \infty$.

Alternative solution: If $\sum b_n < \infty$ we can find a stationary distribution (see part (c)) and since there is a stationary distribution the chain is positive recurrent.

(c) the equations $\pi = \pi P$ give that for each $n \geq 1$ we have $\pi_n = a_{n-1}\pi_{n-1}$. This means that $\pi_1 = a_0 \pi_0$ and $\pi_2 = a_0 a_1 \pi_0 = b_1 \pi_0$, and by induction $\pi_n = b_{n-1}\pi_0$. We can normalize these to have $\sum \pi_n = 1$, which means $\pi_0 = \left(1 + \sum_{n=0}^{\infty} b_n\right)^{-1}$. This can be done if and only if the sum is finite.

Note: For $\pi = \pi P$ if all the equations except one ae satisfied, the last one holds as well automatically, so we do not need to check $\pi_0 = \sum \pi_n P_{n0}$.

Problem 5. Maine (the person, not the state) has 4 umbrellas, some at the office and some at home. Every time she commutes, if it rains and there is an umbrella by her, she takes one. If all the umbrellas are at the other location, she gets wet. Suppose each morning and afternoon it rains with some probability q, independently of all other times. Let X_n be the number of umbrellas at Maine's current location after n trips. For example, if $X_n = 3$ there is one umbrella at the other location. If it is raining, she takes one and $X_{n+1} = 2$ otherwise $X_{n+1} = 1$.

(a) Find the stationary distribution for X_n.

(b) Find the value of q for which the probability of getting wet on an average trip is maximized.

Solution.

(a) With states in order $0, 4, 1, 3, 2$ we get the transition matrix:

$$\begin{pmatrix} 0 & 1 & 0 & 0 & 0 \\ 1-q & 0 & q & 0 & 0 \\ 0 & q & 0 & 1-q & 0 \\ 0 & 0 & 1-q & 0 & q \\ 0 & 0 & 0 & q & 1-q \end{pmatrix}.$$

For example, if there are 3 umbrellas in the current location and it is raining, Maine takes one and has 2 when she arrives at the other location.
This is reversible with equations

$$\pi_0 = \pi_4(1-q) \qquad \pi_4 q = \pi_1 q \qquad \pi_1(1-q) = \pi_3(1-q) \qquad \pi_3 q = \pi_2 q.$$

(All others equations are trivial.) This implies $\pi_1 = \pi_2 = \pi_3 = \pi_4$, and if they add up to 1 we get $\pi = \left(\frac{1-q}{5-q}, \frac{1}{5-q}, \frac{1}{5-q}, \frac{1}{5-q}, \frac{1}{5-q}\right)$.

(b) Maine gets wet with probability $q\pi_0 = \frac{q(1-q)}{5-q}$. Taking derivatives, we find this is maximized at $q = 5 - \sqrt{20} \approx 0.53$, and the probability is $9 - \sqrt{80} \approx 0.056$.