Stochastic Processes
Assignment 4 solutions

Problem 1. Is it possible for a branching process to be reversible? What can be said about ξ in that case (The number of children of each individual is an independent copy of ξ.)

Solution. Yes, but the only such case is if $P(\xi = 1) = 1$ (so that each generation is equal to the previous). This is reversible since $P_{ij} = 0$ unless $i = j$, and any π will work.

To see that this is the only case, note that if $P(\xi = 0) = q > 0$ then $P_{i0} = q^i$ but $P_{0i} = 0$, so the process is not reversible. If $\xi \geq 1$ always, and is sometimes more than 1, then $P_{ij} = 0$ if $i > j$ but can be non-zero if $j > i$, so again it is not reversible. This leaves the case $\xi = 1$ always.

Problem 2. Find the probability generating function $G(s) = \mathbb{E}s^X$ for the following distributions:

(a) $X = \text{Bernoulli}(p)$: here $P(X = 1) = p$ and $P(X = 0) = 1 - p$.
(b) $X = \text{Bin}(n, p)$.
(c) $X = \text{Poi}(\lambda)$.
(d) $X = \text{Geom}(p)$, so $P(X = n) = p(1 - p)^{n-1}$ for $n = 1, 2, \ldots$.

Solution.

(a) $G(s) = ps + (1 - p)$.
(b) $G(s) = (ps + 1 - p)^n$. This can be computed by writing the sum and using the binomial formula, or by noting that this is a sum of n copies of the Bernoulli variable, so the PGF is the product of those.
(c) $G(s) = \sum_n e^{-\lambda} \frac{\lambda^n}{n!} s^n = e^{-\lambda} e^{\lambda s} = e^{\lambda(s-1)}$.
(d) $G(s) = \sum_{n=1}^{\infty} p(1 - p)^{n-1} s^n = \frac{ps}{1-s(1-p)}$ (By summing the geometric series).

Problem 3. Find the probability generating function $G(s) = \mathbb{E}s^X$ for the following distributions:

(a) $X = A + B$ where $A = \text{Bin}(n, p)$ and $B = \text{Geom}(p)$ are independent.
(b) Let $N = \text{Poi}(\lambda)$ and $Y_1, Y_2, \ldots$ be $\text{Geom}(p)$ for some p, λ and all are independent. Let $X = Y_1 + \cdots + Y_N$.
(c) Let $N = \text{Geom}(q)$ and $Y_1, Y_2, \ldots$ be $\text{Geom}(p)$ for some p, q and all are independent. Let $X = Y_1 + \cdots + Y_N$.

Solution.

(a) This is the product of (b) and (d) of the previous problem: $G(s) = (ps + 1 - p)^n \frac{ps}{1-s(1-p)}$.
(b) This is $G_N(G_Y(s))$, with G_N and G_Y from before:

$$G(s) = \exp\left(\lambda\left(\frac{ps}{1 - s(1-p)} - 1\right)\right).$$

(c) This is $G_N(G_Y(s))$, with G_N and G_Y from before:

$$G(s) = \frac{q\frac{ps}{1-s(1-p)}}{1 - (1-q)\frac{ps}{1-s(1-p)}}.$$

Problem 4. Consider a branching process with offspring distribution ξ with

$$P(\xi = 0) = \alpha \qquad P(\xi = 1) = \beta \qquad P(\xi = 2) = 0 \qquad P(\xi = 3) = 1 - \alpha - \beta.$$

(a) If $\alpha = \beta = 1/3$, find the probability the process becomes extinct.
(b) If we start with 4 individuals instead of 1, find the probability the process becomes extinct.
(c) If α, β are such that $\mathbb{E}(\xi) = 2$, what α and β minimize the probability of extinction (starting with 1 individual)? What α, β maximize that probability?

Solution. Let $\gamma = 1 - \alpha - \beta$. We have $G(s) = \alpha + \beta s + \gamma s^3$. $G(\rho) - \rho = 0$ always has solution 1. Dividing by $\rho - 1$ we find the other solutions are

$$\frac{-\gamma \pm \sqrt{\gamma^2 + 4\gamma\alpha}}{2\gamma} = -\frac{1}{2} \pm \sqrt{1/4 + \alpha/\gamma}.$$

(a) If $\alpha = \beta = 1/3$, the solutions are $1, -1/2 \pm \sqrt{5/4}$. Of these the smallest positive one is $\rho = \sqrt{5/4} - 1/2$.

(b) If we start with 4 individuals instead of 1, the process becomes extinct if process starting with every one of the individuals becomes extinct. These are independent and so the probability is ρ^4, with ρ as above.

(c) $\mathbb{E}(\xi) = 2$ gives $3\alpha + 2\beta = 1$. Since α, β and $1 - \alpha - \beta$ must be non-negative, the possible range of α is $[0, 1/3]$. So the problem is to find the minimum and maximum of ρ with these α, and with $\beta = \frac{1-3\alpha}{2}$ and $\gamma = \frac{1+\alpha}{2}$.
We find that
$$\rho = -\frac{1}{2} + \sqrt{1/4 + \alpha/\gamma} = \rho = -\frac{1}{2} + \sqrt{1/4 + 2\alpha/(1+\alpha)}.$$
$\alpha/(1+\alpha)$ is increasing for $\alpha > 0$, so ρ is also increasing in α for $\alpha > 0$. So the minimum is at $\alpha = 0, \beta = 1/2$ and the maximum is at $\alpha = 1/3, \beta = 0$.

Problem 5. Find transition probabilities on $S = \{0, 1, 2, \dots\}$ so that the stationary distribution is $\mathrm{Poi}(\lambda)$ and the only transitions that can hove non-zero probability are $P_{n,n-1}, P_{n,n+1}$, and $P_{n,n}$ for all n.

Solution. Using reversibility we want $\pi_n P_{n,n+1} = \pi_{n+1} P_{n+1,n}$. Using the definition of π_n this simplifies to

$$(n+1)P_{n,n+1} = \lambda P_{n+1,n}.$$

We also need to have $P_{n,n+1} + P_{n,n-1} \le 1$, so that we complete to 1 with $P_{n,n}$.

There are many solutions to these equations.

One which works for $\lambda \ge 1$ is

$$P_{n,n+1} = \frac{1}{2(n+1)} \qquad P_{n+1,n} = \frac{1}{2\lambda},$$

so that all transitions are at most 1/2. For $\lambda \le 1$ we can instead take

$$P_{n,n+1} = \frac{\lambda}{2(n+1)} \qquad P_{n+1,n} = \frac{1}{2}.$$

More generally, you can take a sequence A_n and define the transitions by $(n+1)P_{n,n+1} = \lambda P_{n+1,n} = A_n$. This works as long as $\frac{A_n}{n+1} + \frac{A_{n-1}}{\lambda} \le 1$ for every n. The previous was to take $A_n = \frac{1}{2}\min(1, \lambda)$.