Stochastic Processes

Assignment 5 solutions

Problem 1. Suppose X is an exponential random variavle with unknown rate λ.

(a) Find a for which $P(a \le X \le 2a)$ is maximized.

(b) If $P(X \le 5) = \frac{2}{3}$, find $P(X \ge 10)$, and find λ.

Solution.

(a) We have $f(a) := P(a \le X \le 2a) = e^{-\lambda a} - e^{-2\lambda a}$. Taking derivative w.r.t. a we get $f'(a) = -\lambda e^{-\lambda a} + 2\lambda e^{-2\lambda a}$. This is 0 when $e^{-\lambda a} = 2e^{-2\lambda a}$, or $e^{\lambda a} = 2$, so $a = (\log 2)/\lambda$.

(b) We have $P(X \ge 5) = \frac{1}{3}$. By the memory less property, $P(X \ge 10 | X \ge 5) = P(X \ge 5)$, so $P(X \ge 10) = \frac{1}{9}$.
Note: It is also possible to find λ using $e^{-5\lambda} = \frac{1}{3}$.

Problem 2. Suppose X, Y are independent exponential variables with rates λ, μ respectively.

(a) Prove that $P(X \le Y) = \frac{\lambda}{\lambda+\mu}$.

(b) Calculate the conditional distribution of X given that $X \le Y$.

Solution.

(a) We have

$$\begin{aligned} P(X \le Y) &= \int_0^\infty \int_x^\infty \lambda e^{-\lambda x} \mu e^{-\mu y} dy dx \\ &= \int_0^\infty \lambda e^{-\lambda x} e^{-\mu x} dx \\ &= \frac{\lambda}{\lambda + \mu}. \end{aligned}$$

(b) The density of X conditioned on $X \le Y$ is

$$\frac{1}{P(X \le Y)} \int_x^\infty \lambda e^{-\lambda x} \mu e^{-\mu y} dy dx = (\lambda + \mu) e^{-(\lambda+\mu)x},$$

So conditioned on $X \le Y$ we have that X is just an exponential with rate $\mu + \lambda$.

Problem 3. Recall the bank example from class. Suppose service at teller A takes $\mathrm{Exp}(\lambda_a)$ time and service at teller B takes $\mathrm{Exp}(\lambda_b)$ for some λ_a, λ_b. Naine arrives at the bank when the two tellers are serving customers, but no others are in line. Show that the probability that Naine leaves the bank after both previous customers is $2\frac{\lambda_a \lambda_b}{(\lambda_a+\lambda_b)^2}$.

Hint: use the previous problem.

Solution. Let E be the event that teller A frees up first. From the previous problem, $P(E) = \frac{\lambda_a}{\lambda a + \lambda b}$. If this happens, Naine goes to teller A, and the probability that they leave after teller B finishes is $\frac{\lambda_b}{\lambda a + \lambda b}$. So $P(E, \text{ Naine last}) = P(E)P(\text{Naine last}|E) = \frac{\lambda_a \lambda_b}{(\lambda_a+\lambda_b)^2}$.

Similarly, $P(E^c, \text{ Naine last})$ is the same and together the claim follows.

Problem 4. Oaine is having office hours and Paine and Qaine will show up. The arrival times are independent exponentials with rates λ_P and λ_Q. After arriving they stay for exponential times with rates μ_P and μ_Q respectively (all independent).

(a) What is the probability that Paine comes and leaves before Qaine arrives?

(b) What is the expected time for the last student to arrive?

(c) [optional] What is the expected time for the last student to leave?

Solution.

(a) We need Paine to come first, which has probability $\frac{\lambda_P}{\lambda P+\lambda_Q}$ and then to leave before Qaine arrives, with probability $\frac{\mu_P}{\mu P+\lambda_Q}$. The product is the answer.

(b) The probability that both arrive by time t is $(1-e^{-\lambda_P t})(1-e^{-\lambda_Q t})$, so the probability that the last arrival is after time t is $P(X \geq t) = e^{-\lambda_P t} + e^{-\lambda_Q t} - e^{-(\lambda_Q+\lambda_Q)t}$.
Using $E(X) = \int_0^\infty P(X \geq t)$ we get

$$E(X) = \frac{1}{\lambda_P} + \frac{1}{\lambda_Q} - \frac{1}{\lambda_P + \lambda_Q}.$$

More clever solution: If the arrival times are X, Y then use

$$X + Y = \min(X,Y) + \max(X,Y).$$

The minimum is $\mathrm{Exp}(\lambda_P + \lambda_Q)$, with expectation $\frac{1}{\lambda_P+\lambda_Q}$. Therefore

$$E\max(X,Y) = E(X) + E(Y) - E\min(X,Y) = \frac{1}{\lambda_P} + \frac{1}{\lambda_Q} - \frac{1}{\lambda_P + \lambda_Q}.$$

(c) The probablity for a student leaving after time t is found to be

$$\begin{aligned} e^{-\lambda t} + \int_0^t \lambda e^{-\lambda x} P(\text{stay } \geq (t-x))dx &= e^{-\lambda t} + \int_0^t \lambda e^{-\lambda x} e^{-\mu(t-x)} dx \\ &= \frac{\lambda e^{\mu t} - \mu e^{-\lambda t}}{\lambda - \mu}. \end{aligned}$$

With subscript P or Q this works for the two students, giving a formula for the probability that the last departure is after time t. Integrate this from 0 to infinity to find the expectation.

Problem 5. For a Poisson process $N(t)$ with rate λ, find the following:
(a) $P(N(t) = n | N(s) = m)$ for $m \leq n$ and $s \leq t$.
(b) $P(N(s) = m | N(t) = n)$ for $m \leq n$ and $s \leq t$.

Solution. We have

$$P(N(s) = m) = e^{-\lambda s}\frac{(\lambda s)^m}{m!} \qquad P(N(t) = n) = e^{-\lambda t}\frac{(\lambda t)^n}{n!}.$$

The probability of the intersection is determined by independent increments on $(0, s)$ and (s, t):

$$P(N(s) = m, N(t) = n) = P(N(s) = m)P(N(t) - N(s) = n - m) = e^{-\lambda s}\frac{(\lambda s)^m}{m!} \cdot e^{-\lambda(t-s)}\frac{(\lambda(t-s))^(n-m)}{(n-m)!}.$$

(a) This comes to $P(N(t) - N(s) = n - m) = e^{-\lambda(t-s)}\frac{(\lambda(t-s))^(n-m)}{(n-m)!}$.

(b) This simplifies to

$$\frac{n! s^m (t-s)^{n-m}}{m!(n-m)! t^n}.$$

This is the probability that $\mathrm{Bin}(n, s/t) = m$, which can be deduced also from the theorem that the n events are uniform on $[0, t]$, so each is in $[0, s]$ eith probability s/t.