Stochastic Processes

Assignment 6 solutions

Problem 1. In a pizzeria the orders arrive according to a Poisson process of rate λ. Given that 3 orders arrived in an hour, determine the conditional probability density functions of the arrival of the first times S_1, S_2, S_3. (Note: this asks for 3 probability density functions, not the joint p.d.f.)

Solution. The three times are distributed as the order statistics $Y(1), Y(2), Y(3)$ for three uniform samples in $[0, 1]$.

- The minimal has $P(S_1 \leq t) = 1 - (1-t)^3$ so the p.d.f. is $3(1-t)^2$.
- The middle event has $P(S_2 \leq t) = t^3 + 3t^2(1-t)$ so the p.d.f. is $3t^2 + 6t(1-t) - 3t^2 = 6t(1-t)$.
- The maximal has $P(S_3 \leq t) = t^3$ so the p.d.f. is $3t^2$.

In general, let $Y(k)$ be the kth order statistic out of n uniform in $[0,1]$. Then $Y(k) \in (t, t+\varepsilon)$ requires $k-1$ to be in $(0,t)$ and $n-k$ to be in $(t,1)$, and one to be in $(t, t+\varepsilon)$. This has probability $\frac{n!}{(k-1)!(n-k)!} t^{k-1}(1-t)^{n-k}\varepsilon$, so the p.d.f. is $\frac{n!}{(k-1)!(n-k)!} t^{k-1}(1-t)^{n-k}$. This is a Beta distribution.

Problem 2. Let $N(t)$ be a Poisson process of rate λ. Determine the conditional distribution of $N(3)$ given that $N(5) = 7$.

Solution. Each of the 7 events up to time 5 has probability $3/5$ to be before time 3, and all are independent, so the conditional distribution is $\text{Bin}(7, 3/5)$.

Problem 3. Raine is a birdwatcher and counts the birds in her garden, which arrive according to a Poisson process of rate λ. If an hour passes with no birds, she gets bored and goes to make tea.

(a) What is the probability that she doesn't see any birds?
(b) How much time on average does she spend birdwatching? Hint: Condition on the time of the first observed bird.
(c) What is the distribution of the number of birds seen?

Solution.

(a) This is $e^{-\lambda}$: the probability that a waiting time is more than 1.
(b) With probability $e^{-\lambda}$ she spends one hour and stops. Otherwise, she spends X_1 and the boredom clock is reset. If $X_1 = x$, then the total time is on average $x + E(T)$. Therefore the wanted time T has:

$$E(T) = e^{-\lambda} \cdot 1 + \int_0^1 \lambda e^{-\lambda x}(x + E(T))dx$$

This simplifies to

$$E(T) = \frac{1 - e^{-\lambda}}{\lambda} + (1 - e^{-\lambda})E(T),$$

from which we find

$$E(T) = \frac{1 - e^{-\lambda}}{\lambda e^{-\lambda}}.$$

(c) Each time she gives up before seeing another bird with probability $1 - e^{-\lambda}$, so the number of birds seen is $\text{Geom}(e^{-\lambda})$ minus 1:

$$p_k = e^{-\lambda}(1 - e^{-\lambda})^k.$$

Problem 4. Players arrive at a casino at rate λ, and begin betting, and losing money at rate \$1/minute. Let X be the total amount they lose by time t.

(a) What is $E(X|N(t))$?
(b) Explain why $\text{Var}(X|N(t)) = N(t)t^2/12$.
(c) Find $\text{Var}(X)$.

Solution.

(a) Given that $N(t) = k$, there are k arrivals. Each is uniformly distributed in $[0, t]$, so the average profit from each is $t/2$, and so $E(X|N(t)) = N(t)\frac{t}{2}$.

(b) The variance of the profit from each player is the variance of a random variable uniform in $[0, t]$, which is $t^2/12$. Since these are independent, the variance of the sum is the sum of variances.

(c) We use $N = N(t)$ for clarity. We use the formula

$$\mathrm{Var}(X) = E(\mathrm{Var}(X|N)) + \mathrm{Var}(E(X|N)).$$

The first term is

$$E(Nt^2/12) = E(N) \cdot t^2/12 = \lambda t \cdot t^2/12.$$

The second is

$$\mathrm{Var}(Nt/2) = (t/2)^2 \mathrm{Var}(N) = (t/2)^2 \cdot \lambda t.$$

Together we get

$$\mathrm{Var}(X) = (\lambda t^3)(1/12 + 1/4) = \frac{\lambda t^3}{3}.$$

Problem 5. A system suffers faults by a Poisson process with rate λ. Each fault will, independently, cause the system to fail after time X with c.d.f. F. When the system fails, the responsible fault is fixed, but earlier other faults may remain and cause later failure.

(a) What is the distribution of the number of failures up to t?

(b) What is the distribution of the number of faults that remain in the system at time t and are not yet fixed?

(c) Are the random variables in parts (a) and (b) dependent or independent? (justify!)

Solution.

(a) A fault at time s causes a failure before time t (call this type 1) with probability $F(t-s)$. The number of failures up to time t is Poisson with mean $\int_0^t \lambda F(t-s)ds$.

(b) A fault at time s has not caused a failure before time t (call this type 2) with probability $1 - F(t-s)$. The number of failures up to time t is Poisson with mean $\int_0^t \lambda(1 - F(t-s))ds$.

(c) By the theorem from class, these are independent.