Stochastic Processes

Assignment 7 solutions

Note: Start each problem on a **new page**.

Problem 1. Potential customers arrive at a queue at times of a Poisson process with rate λ. If a customer sees n others in the queue, the join with probability α_n, and give up and go home with probability $1-\alpha_n$. The service rate is always μ. Set this up as a birth and death process and determine the birth and death rates.

Solution. The birth rate λ_n from n to $n+1$ is $\lambda \cdot \alpha_n$, since in a small time interval h there is probability $\lambda h + o(h)$ that someone arrives, and probability α_n that someone who arrives decides to join the queue. The death rate is $\mu_n = \mu$ since that is always the rate at which people are beining served.

Problem 2. Consider two machines, both of which have an exponential lifetime with parameter λ. There is a single repairman can service machines at an exponential rate μ. Let X_t be the number of machines that are operational at time t. Write the Kolmogorov backward equations (you need not solve them).

Solution. The state space is $\{0, 1, 2\}$. From 2 we transition to 1 at rate 2λ, since there are 2 machines that might break at rate λ each. From 1, we transition to 0 at rate λ, and to 2 at rate μ. From 0 we transition to 1 at rate μ. In all we have

$$q_{01} = \mu \qquad q_{10} = \lambda \qquad q_{12} = \mu \qquad q_{21} = 2\lambda,$$

Other transition rates are 0.

Problem 3. Consider a Yule process with parameters $\lambda_n = n\lambda$ and $\mu_n = 0$. Start the process with $X_0 = 1$ individual. Let T_n be the time at which the population reaches size n (so $T_1 = 0$).

(a) Explain why $T_{n+1} - T_n$ are independent exponential variables, and give their parameters.

(b) Find $E(T_n)$. (You do not need to simplify sums.)

Solution.

(a) From n the only jump is to $n+1$, after $\text{Exp}(v_n)$ time. These are independent because of the Markov property.

(b) $T_n = \sum_{i=1}^{n-1} T_{i+1} - T_i$ with $T_1 = 0$. The expectation is the sum of expectations:

$$E(T_n) = \sum_{i=1}^{n-1} \frac{1}{v_i} = \sum_{i=1}^{n-1} \frac{1}{\lambda i} \approx \lambda^{-1} \log n.$$

Problem 4. A continuous time Markov chain on states $\{a, b, c\}$ has jump rates

$$\begin{array}{ccc} q_{ab} = 1 & q_{ba} = 2 & q_{ca} = 2 \\ q_{ac} = 1 & q_{bc} = 1 & q_{cb} = 2. \end{array}$$

Suppose $X_0 = a$. We wish to find the expected time to reach state b, denoted T_b.

(a) What is the expected time for the first jump out of a?

(b) Once the chain leaves a, what is the distribution of the next state?

(c) Let M_i be the expected time to reach b if we start at state i. (So $M_b = 0$). Use parts (a),(b) to write an equation for M_a in terms of M_b and M_c.

(d) Write a similar equation for M_c.

(e) Solve the equations to find M.

Solution.

(a) This is Exp(2) so average is 1/2.

(b) After leaving a the chain is at b or c with equal probability 1/2.

(c) From a we have

$$M_a = \tfrac{1}{2} + \tfrac{1}{2}M_b + \tfrac{1}{2}M_c.$$

(d) From c we hav similarly

$$M_c = \tfrac{1}{4} + \tfrac{1}{2}M_b + \tfrac{1}{2}M_a.$$

(e) $M_a = 5/6$ and $M_v = 2/3$.

Problem 5 (Ross, 6.2)**.** A one-celled organism can be in one of two states: A or B. An individual in state A will change to state B at an exponential rate α. An individual in state B divides into two new individuals of type A at an exponential rate β. Define an appropriate continuous-time Markov chain for a population of such organisms and determine the appropriate parameters for this model (the transition rates and jump probabilities.) (Hint: the state (N_A, N_B) is the number of individuals of each type.)

Solution. Let (a, b) be the state with a individuals of type A and b of type B.

- The total rate of changes A→B is αa, so thats the rate of jumping to $(a-1, b+1)$. The total
- At rate βb we jump from (a, b) to $(a+2, b-1)$.
- The overall jump rate is $v_{(a,b)} = \alpha a + \beta b$.
- The probability of jumping to $(a-1, b+1)$ is $\frac{\alpha a}{\alpha a + \beta b}$. The probability of jumping to $(a+2, b-1)$ is the complement.