Stochastic Processes

Assignment 8 solutions

Note: Start each problem on a **new page**.

Problem 1. Consider the Yule process: a pure birth chain, where the rate of jumping from n to $n+1$ is λn. Suppose $X_0 = 1$.

(a) Write down the backward Kolmogorov equations for $P_{ij}(t)$.
(b) Use these to find $P_{11}(t)$.
(c) Use these to find $P_{12}(t)$.

Solution.

(a) The general equation is $P'_{ij}(t) = \sum_{k \neq j} q_{ik} P_{kj}(t) - v_i P_{ij}(t)$. The first term is zero unless $k = i + 1$ (that's the only jump allowed), so this becomes

$$P'_{ij}(t) = \lambda i P_{i+1,j}(t) - \lambda i P_{ij}(t).$$

(b) For $i = j = 1$ we get

$$P'_{11}(t) = \lambda P_{21}(t) - \lambda P_{11}(t).$$

However, it is impossible to get from 2 to 1, so the first term is 0, and the solution is $P_{11}(t) = e^{-\lambda t}$. (This is also proved by noting that this is the probability to not jump by time t, and that's the only way to be at 1 at time t.

(c) For $i = 1, j = 2$ this gives

$$P'_{12}(t) = \lambda P_{22}(t) - \lambda P_{12}(t).$$

As in the previous part we have $P_{22}(t) = e^{-2\lambda t}$. The solution to $f'(x) = \lambda e^{2x} - \lambda f'(x)$ is

$$f(x) = e^{-2\lambda x} + Ce^{-x}.$$

Since $P_{12}(0) = 0$ we must have $C = -1$ and get $P_{12}(t) = e^{-\lambda t} - e^{-2\lambda t}$.

Problem 2. A factory has three machines. Each breaks down at rate 1. If any machine is broken, a repairman works to fix it, and fixes at rate 2. Let X_t be the number of operational machines at time t. What is the limit probability for having no working machine? What is the limit probability for having all machines working?

Solution. The states are $0, 1, 2$. The transition rates are

$$q_{21} = 2 \qquad q_{10} = 1 \qquad q_{01} = 2 \qquad q_{12} = 2.$$

(I will also accept the interpretation with $q_{01} = 4$.)

The equations for the limit probabilities are

$$2P_0 = P_1 \qquad 3P_1 = 2P_0 + 2P_2 \qquad 2P_2 = 2P_1,$$

and $P_0 + P_1 + P_2 = 1$. The solution is $P_0 = 1/5$, $P_1 = P_2 = 2/5$.

Problem 3. (Ross: 6.32): Customers arrive at a two-server station in accordance with a Poisson process having rate λ. Upon arriving, they join a single queue. Whenever a server completes a service, the person first in line enters service. The service times of server i are exponential with rate μ_i, $i = 1, 2$, where $u_1 + \mu_2 > \lambda$. An arrival finding both servers free is equally likely to go to either one. Define an appropriate continuous-time Markov chain for this model, show it is time reversible, and find the limiting probabilities.

Solution. Let X_t be the total number of customers in the queue (including those being serviced). If there is only one being serviced, then we also need to know which server they are with, so the states are $\{0, a = 1_1, b = 1_2, 2, 3, 4, \dots,$ with 1_i meaning the single customer is with server i. The Transition rates are as follows. Arrivals give

$$q_{0,a} = q_{0,b} = \lambda/2, \qquad q_{a,2} = q_{b,2} = q_{n,n+1} = \lambda.$$

Services give

$$q_{a,0} = q_{2,b} = \mu_1 \qquad q_{b,0} = q_{2,a} = \mu_2, \qquad q_{n+1,n} = \mu_1 + \mu_2.$$

($n \geq 2$ can be anything.) For reversibility with limit probabilities P_i we need to have $P_i q_{ij} = P_j q_{ji}$ for all pairs of states. The non-zero equations are

$$P_{n+1}(\mu_1 + \mu_2) = P_n \lambda \text{ for } n \geq 2,$$

and

$$P_0\lambda/2 = P_a\mu_1 \qquad P_0\lambda/2 = P_b\mu_2 \qquad P_b\lambda = P_2\mu_1 \qquad P_a\lambda = P_2\mu_2.$$

These are solved by

$$P_a = \frac{\lambda}{2\mu_1}P_0 \qquad P_b = \frac{\lambda}{2\mu_2}P_0 \qquad P_2 = \frac{\lambda^2}{2\mu_1\mu_2}P_0 \qquad P_n = \left(\frac{\lambda}{\mu_1+\mu_2}\right)^{n-2} P_2$$

for $n \geq 2$.

These have sum $P_0\left(1 + \frac{\lambda}{2\mu_1} + \frac{\lambda}{2\mu_2} + \frac{\lambda^2}{2\mu_1\mu_2(1-\lambda/(\mu_1+\mu_2))}\right)$, which determines P_0.

Problem 4. (Ross: 6.33) Consider two M/M/1 queues with respective parameters λ_i, μ_i for $i = 1, 2$. Suppose they share a common waiting room that can hold at most three customers. That is, whenever an arrival finds her server busy and three customers in the waiting room, she goes away. Find the limiting probability that there will be n queue 1 customers and m queue 2 customers in the system.

Solution. This problem is unclear and has several interpretations.. Let (a, b) be the state with i customers for server 1 and b for server 2. The possible states are (a, b) with $a + b \leq 3$. There are 10 states overall. The transitions: Increase a at rate λ_1 if possible. Increase b at rate λ_2 if possible. Decrease a at rate μ_1 if possible. Decrease b at rate μ_2 if possible.

The equations for the limit probabilities:

$$\begin{aligned}
P_{00} &= \mu_1 P_{10} + \mu_2 P_{01} \\
P_{01} &= \lambda_2 P_{00} + \mu_1 P_{11} + \mu_2 P_{02} \\
P_{02} &= \lambda_2 P_{01} + \mu_1 P_{12} + \mu_2 P_{03} \\
P_{03} &= \lambda_2 P_{02} \\
P_{10} &= \lambda_1 P_{00} + \mu_1 P_{20} + \mu_2 P_{11} \\
P_{11} &= \lambda_1 P_{01} + \lambda_2 P_{10} + \mu_1 P_{21} + \mu_2 P_{12} \\
P_{12} &= \lambda_1 P_{02} + \lambda_2 P_{11} \\
P_{20} &= \lambda_1 P_{10} + \mu_1 P_{30} + \mu_2 P_{21} \\
P_{21} &= \lambda_1 P_{11} + \lambda_2 P_{20} \\
P_{30} &= \lambda_1 P_{20}
\end{aligned}$$

These can be solved though it is messy. It is much easier if we guess reversibility. In that case we find that $P_{ab} = \left(\frac{\lambda_1}{\mu_1}\right)^a \left(\frac{\lambda_2}{\mu_2}\right)^b$ is a solution to detailed balance, and it can be normalized to have sum 1.

Problem 5. A factory has N identical machines. When a machine breaks down, its operator immediately begins to repair it. Each machine breaks down at rate μ, and each repair independently takes an exponential time of rate λ. Let $X(t)$ denote the number of machines that are working at time t. This defines a birth and death process.

(a) Determine the birth and death rates.
(b) Determine the limiting probabilities. (It is a certain binomial distribution.)
(c) Suppose that $N = 50$, $\lambda = 10$, $\mu = 1$. What is the average number of machines that are operating, in the long run?

Solution.

(a) The birth rate is $\lambda_n = \lambda(N - n)$.
(b) The death rate is $\mu_n = \mu n$.
(c) Detailed balance gives $P_{n+1} = P_n \frac{\lambda_n}{\mu_{n+1}}$, so

$$P_n = P_0 \frac{\lambda_0 \dots \lambda_{n-1}}{\mu_1 \dots \mu_n} = P_0 \frac{\lambda^n N(N-1)\dots(N-n+1)}{\mu^n n!} = P_0 \binom{N}{n} \frac{\lambda^n}{\mu^n}.$$

From this we deduce that $P_0 = (1 + \mu/\lambda)^{-N}$, and hence

$$P_n = \binom{N}{n} a^n (1-a)^{N-n}$$

with $a = \lambda/(\lambda + \mu)$. This is $\mathrm{Bin}(N, a)$.

(d) Here the limit probabilities are $\mathrm{Bin}(50, 10/11)$, so the average number of working machines is $500/11$.