Stochastic Processes

Assignment 9 solutions

Note: Start each problem on a **new page**.

Problem 1. We consider a model for a rumour (or a disease) spreading. The population has N individuals. Initially only one knows the rumour. Each pair of individuals meet at rate 1. When that happens, if one knows the rumour they tell the other.

(a) How many ways are there to go from k individuals knowing the rumour to $k+1$?
(b) Let X_t be the number of people who know the rumour at time t. Set this up as a continuous time Markov chain, and give the transition rates.
(c) Let T_k be the time at which the chain reaches k. What is the distribution of $T_{k+1} - T_k$?
(d) Write an expression for $E(T_N)$: the time it takes for everyone to know the rumour. (You can leave this as a sum.)

Solution.

(a) $k(N-k)$ pairs of indviduals where one knows the rumour and the other does not.
(b) X_t is a pure birth chain with rate $\lambda_k = k(N-k)$.
(c) This is $E\,\mathrm{Exp}(v_k) = \frac{1}{k(N-k)}$.
(d) $T_N = \sum T_i - T_{i-1}$, and so $E(T_N) = \sum_{i=1}^{N-1} \frac{1}{i(N-i)}$.

Problem 2. Consider the birth and death chain on $\{0, 1, 2, 3, 4\}$ with jump rates $\lambda_i = 4 - i$ and $\mu_i = 2$. Suppose $X_0 = 2$.

(a) We wish to find the probability that the chain reaches state 4 before reaching state 0. Let u_i be the probability of reaching state 4 before state 0 if the chain starts at state i. Write equations for u_i in terms of u_{i-1} and u_{i+1}. (Hint: condition on the first jum of the chain.)
(b) Solve these equations to determine u_i for every i.
(c) Next, we wish to find the expected time to reach state 4. Let s_i be the expected time to reach state 4 if we start at state i. We can write that time a the time it takes to make the first jump, plus the time to reach 4 from whatever state the jump was to. Use this to write equations relating s_i to s_{i+1} and s_{i-1}.
(d) Solve these equations.

Solution.

(a) We have $u_4 = 1$ and $u_0 = 0$ (trivially). For the other states we have

$$u_i = P_{i,i+1}u_{i+1} + P_{i,i-1}u_{i-1},$$

where

$$P_{i,i+1} = \frac{\lambda_i}{\lambda_i + \mu_i} = \frac{4-i}{6-i} \qquad P_{i,i-1} = \frac{\mu_i}{\lambda_i + \mu_i} = \frac{2}{6-i}.$$

(b) The solution to these is

$$u_1 = \frac{3}{8} \qquad u_2 = \frac{5}{8} \qquad u_3 = \frac{7}{8}.$$

(c) Here, the average time to state 4 is the average time for the first jump: $1/v_i$ plus the time from the next state the chain jumps to. We have $s_4 = 0$ (trivially) and $s_0 = \frac{1}{4} + s_1$, and for the others

$$s_i = \frac{1}{6-i} + P_{i,i+1}s_{i+1} + P_{i,i-1}s_{i-1}.$$

(d) The solution is

$$s_0 = \frac{19}{4} \qquad s_1 = \frac{9}{2} \qquad s_2 = 4 \qquad s_3 = 3.$$

Problem 3. Batman is chasing the joker. There are four places each could be, arranged in the corners of a square. Batman is moving from his corner to an adjacent corner at rate 1 (rate 1/2 to each). The Joker picks some rate v, and moves from his corner at rate v, in the same way. Find the expected time before they meet (are at the same corner). Hint: at each time they are either in opposite corners, nearby corners, or the same corner. Find transition probabilities between these.

Solution. Even though there are 16 states, we can consider only 3 types of states given by the distance between them. State 0: they are at the same corner. State 1: adjacent corners. State 2: opposite corners.

The jump rates are as follows: $q_{21} = 1 + v$ since whoever jumps will get to state 1. Similarly, $q_{10} = q_{12} = \frac{1+v}{2}$, since whoever jums is equally likely to get to state 0 or 2. Let s_i be the time to meet starting at state i. As in the previous problem we get

$$s_0 = 0 \qquad s_2 = \frac{1}{1+v} + s_1 \qquad s_1 = \frac{1}{1+v} + \frac{1}{2}s_2 + \frac{1}{2}s_0.$$

The solution to this is $s_2 = \frac{4}{1+v}$ and $s_1 = \frac{3}{1+v}$.

Problem 4. Batman is again chasing the Joker. This time the chase is in discrete time. In each step, Batman stays in his current corner with probability α, and otherwise moves to a random nearby corner. The joker stays in place with probability β and otherwise jumps to a nearby corner.

(a) As in the previous problem, find the transition probabilities for the distance between Batman and the Joker.

(b) If they start at opposite corners, find the expected time to meet. Note: The case $\alpha = \beta = 0$ is different!

Solution.

(a) The main difference is that it is possible that both jump at once, in which case the distance can remain the same or change by 2. If they are in adjacent corners and both jump, they will again be in adjacent corners. If they are in opposite corners, they get to either the same corner, or opposite corners again depending on the directions. The transition probabilities are:

$$\begin{aligned}
&P_{00} = \alpha\beta + (1-\alpha)(1-\beta)/2 && P_{01} = \alpha(1-\beta) + (1-\alpha)\beta && P_{02} = (1-\alpha)(1-\beta)/2 \\
&P_{10} = \frac{1}{2}(\alpha(1-\beta) + (1-\alpha)\beta) && P_{11} = \alpha\beta + (1-\alpha)(1-\beta) && P_{12} = \frac{1}{2}(\alpha(1-\beta) + (1-\alpha)\beta) \\
&P_{20} = (1-\alpha)(1-\beta)/2 && P_{21} = \alpha(1-\beta) + (1-\alpha)\beta && P_{22} = \alpha\beta + (1-\alpha)(1-\beta)/2
\end{aligned}$$

(b) The time to meet from state i satisfies $s_0 = 0$ and

$$s_i = 1 + \sum P_{ij} s_j$$

for $i = 1, 2$. The solution for this is $s_2 = \frac{4}{1-ab}$. (If $a = b = 1$ then $s_2 = \infty$).