Mathematics 318 — Final exam — 2019-04-09

- The test consists of 17 pages and 9 questions worth a total of 100 marks.
- Time allowed: 150 minutes.
- This is a closed-book examination. **None of the following are allowed**: documents, cheat sheets or electronic devices of any kind (including calculators, cell phones, etc.)
- No work on this page or the formula sheet will be marked. Use the reverse side of each page for extra space.
- Fill in the information below **before** starting the midterm.
- Good luck, and don't fuck it up.

Student number								
Family Name								
Preferred Name								
Given Name								

Student conduct during examinations

(a) Each examination candidate must be prepared to produce, upon the request of the invigilator or examiner, his or her UBCcard for identification.
(b) Examination candidates are not permitted to ask questions of the examiners or invigilators, except in cases of supposed errors or ambiguities in examination questions, illegible or missing material, or the like.
(c) No examination candidate shall be permitted to enter the examination room after the expiration of one-half hour from the scheduled starting time, or to leave during the first half hour of the examination. Should the examination run forty-five (45) minutes or less, no examination candidate shall be permitted to enter the examination room once the examination has begun.
(d) Examination candidates must conduct themselves honestly and in accordance with established rules for a given examination, which will be articulated by the examiner or invigilator prior to the examination commencing. Should dishonest behaviour be observed by the examiner(s) or invigilator(s), pleas of accident or forgetfulness shall not be received.
(e) Examination candidates suspected of any of the following, or any other similar practices, may be immediately dismissed from the examination by the examiner/invigilator, and may be subject to disciplinary action:
    (i) speaking or communicating with other examination candidates, unless otherwise authorized;
    (ii) purposely exposing written papers to the view of other examination candidates or imaging devices;
    (iii) purposely viewing the written papers of other examination candidates;
    (iv) using or having visible at the place of writing any books, papers or other memory aid devices other than those authorized by the examiner(s); and,
    (v) using or operating electronic devices including but not limited to telephones, calculators, computers, or similar devices other than those authorized by the examiner(s)(electronic devices other than those authorized by the examiner(s) must be completely powered down if present at the place of writing).
(f) Examination candidates must not destroy or damage any examination material, must hand in all examination papers, and must not take any examination material from the examination room without permission of the examiner or invigilator.
(g) Notwithstanding the above, for any mode of examination that does not fall into the traditional, paper-based method, examination candidates shall adhere to any special rules for conduct as established and articulated by the examiner.
(h) Examination candidates must follow any additional examination rules or directions communicated by the examiner(s) or invigilator(s).

Table 1: Common Distributions

Distribution	Mean	Variance	Characteristic function
Binomial (n, p)	np	$np(1-p)$	$(1-p+pe^{it})^n$
Geometric (p)	$1/p$	$\frac{1-p}{p^2}$	$\frac{pe^{it}}{1-(1-p)e^{it}}$
Poisson (λ)	λ	λ	$e^{\lambda(e^{it}-1)}$
Uniform (a, b)	$\frac{a+b}{2}$	$\frac{(b-a)^2}{12}$	$\frac{e^{ita}-e^{itb}}{it(b-a)}$
Exponential (λ)	$1/\lambda$	$1/\lambda^2$	$\frac{\lambda}{\lambda-it}$
Normal (μ, σ^2)	μ	σ^2	$e^{i\mu t-\sigma^2 t^2/2}$

The normal CDF:

x	0.00	0.01	0.02	0.03	0.04	0.05	0.06	0.07	0.08	0.09
0.0	0.5000	0.5040	0.5080	0.5120	0.5160	0.5199	0.5239	0.5279	0.5319	0.5359
0.1	0.5398	0.5438	0.5478	0.5517	0.5557	0.5596	0.5636	0.5675	0.5714	0.5753
0.2	0.5793	0.5832	0.5871	0.5910	0.5948	0.5987	0.6026	0.6064	0.6103	0.6141
0.3	0.6179	0.6217	0.6255	0.6293	0.6331	0.6368	0.6406	0.6443	0.6480	0.6517
0.4	0.6554	0.6591	0.6628	0.6664	0.6700	0.6736	0.6772	0.6808	0.6844	0.6879
0.5	0.6915	0.6950	0.6985	0.7019	0.7054	0.7088	0.7123	0.7157	0.7190	0.7224
0.6	0.7257	0.7291	0.7324	0.7357	0.7389	0.7422	0.7454	0.7486	0.7517	0.7549
0.7	0.7580	0.7611	0.7642	0.7673	0.7704	0.7734	0.7764	0.7794	0.7823	0.7852
0.8	0.7881	0.7910	0.7939	0.7967	0.7995	0.8023	0.8051	0.8078	0.8106	0.8133
0.9	0.8159	0.8186	0.8212	0.8238	0.8264	0.8289	0.8315	0.8340	0.8365	0.8389
1.0	0.8413	0.8438	0.8461	0.8485	0.8508	0.8531	0.8554	0.8577	0.8599	0.8621
1.1	0.8643	0.8665	0.8686	0.8708	0.8729	0.8749	0.8770	0.8790	0.8810	0.8830
1.2	0.8849	0.8869	0.8888	0.8907	0.8925	0.8944	0.8962	0.8980	0.8997	0.9015
1.3	0.9032	0.9049	0.9066	0.9082	0.9099	0.9115	0.9131	0.9147	0.9162	0.9177
1.4	0.9192	0.9207	0.9222	0.9236	0.9251	0.9265	0.9279	0.9292	0.9306	0.9319
1.5	0.9332	0.9345	0.9357	0.9370	0.9382	0.9394	0.9406	0.9418	0.9429	0.9441
1.6	0.9452	0.9463	0.9474	0.9484	0.9495	0.9505	0.9515	0.9525	0.9535	0.9545
1.7	0.9554	0.9564	0.9573	0.9582	0.9591	0.9599	0.9608	0.9616	0.9625	0.9633
1.8	0.9641	0.9649	0.9656	0.9664	0.9671	0.9678	0.9686	0.9693	0.9699	0.9706
1.9	0.9713	0.9719	0.9726	0.9732	0.9738	0.9744	0.9750	0.9756	0.9761	0.9767
2.0	0.9772	0.9778	0.9783	0.9788	0.9793	0.9798	0.9803	0.9808	0.9812	0.9817
2.1	0.9821	0.9826	0.9830	0.9834	0.9838	0.9842	0.9846	0.9850	0.9854	0.9857
2.2	0.9861	0.9864	0.9868	0.9871	0.9875	0.9878	0.9881	0.9884	0.9887	0.9890
2.3	0.9893	0.9896	0.9898	0.9901	0.9904	0.9906	0.9909	0.9911	0.9913	0.9916
2.4	0.9918	0.9920	0.9922	0.9925	0.9927	0.9929	0.9931	0.9932	0.9934	0.9936
2.5	0.9938	0.9940	0.9941	0.9943	0.9945	0.9946	0.9948	0.9949	0.9951	0.9952
2.6	0.9953	0.9955	0.9956	0.9957	0.9959	0.9960	0.9961	0.9962	0.9963	0.9964
2.7	0.9965	0.9966	0.9967	0.9968	0.9969	0.9970	0.9971	0.9972	0.9973	0.9974
2.8	0.9974	0.9975	0.9976	0.9977	0.9977	0.9978	0.9979	0.9979	0.9980	0.9981
2.9	0.9981	0.9982	0.9982	0.9983	0.9984	0.9984	0.9985	0.9985	0.9986	0.9986
3.0	0.9987	0.9987	0.9987	0.9988	0.9988	0.9989	0.9989	0.9989	0.9990	0.9990

1. [9 marks] A random variable X has p.d.f. $\frac{x+2}{4}$ on $[-1,1]$ and 0 outside $[-1,1]$.

   (a) What is the expectation $\mathbf{E}X$?

   (b) What is the variance $\operatorname{Var} X$?

   (c) What is the characteristic function of X?

2. [14 marks] A committee of 7 people is chosen randomly out of a group of 60. Of the 60, Dorothy has 20 friends and 15 enemies. Let X be the number of friends of Dorothy in the committee and Y the number of enemies.

   (a) Find $P(X \geq 5)$, and $P(Y \geq 5)$.

   (b) Find $P(\{X \geq 5\} \cup \{Y \geq 5\})$?

(c) What is $E(X - Y)$?

(d) What is $E(X|Y = y)$?

3. [14 marks] Let X, Y be independent exponential variables with parameters λ. Compute all integrals in the following.

   (a) Find $E[XY]$.

   (b) Find $P(X > Y + 2)$.

(c) Find the density function of $X - Y$.

(d) Find the characteristic function of $X - Y$.

4. [12 marks] Customers arrive at a store according to a Poisson process with rate $\lambda = 6$ per hour. The store is open from 8:00 to 18:00.

   (a) What is the probability that at least two customers arrive between 8:00 and 8:30?

   (b) Use the Central limit theorem to estimate the probability that at most 50 customers come in a day. You may use Φ in your answer.

(c) What is the expected number of customers arriving before noon?

(d) What is the expected number of customers arriving all day, conditioned on 30 customers arriving before noon?

5. [8 marks] Jack and Jill are magicians, who perform the following trick. A volunteer picks a card, and Jack or Jill makes a prediction what the card is.

   (a) Jack is not very talented, and the trick succeeds only 1% of the time. Use the Poisson approximation to estimate the probability that the trick succeeds exactly three times out of 200 attempts.

   (b) Jill succeeds at the same trick 200 out of 1000 independent attempts, give a 95% confidence interval for the probability that she succeeds at each attempt.

6. [12 marks] Biking to work takes Martha a normal $N(23, 2^2)$ number of minutes. The subway takes a fixed 15 minutes, plus the random waiting time for the train which is Exp(0.2). Martha takes the bus with probability 1/3 each day. If the trip takes more than 25 minutes, she is late.

   (a) What is the probability that Martha is late for work on any day?

   (b) What is the probability she took the bus, conditioned on her being late?

(c) What is the Variance of Martha's travel time?

7. [7 marks] A random variable X has characteristic function $\phi(t) = \frac{\cos(t)}{1+t^2}$. Compute the following: (a) $E[X]$. (b) $\mathrm{Var}(X)$. (c) $E[X^4]$.

8. [16 marks] Consider a Markov chain with states $\{1, 2, 3, 4, 5, 6\}$ and the following transition probability matrix:

$$\begin{pmatrix} 0 & 1/2 & 1/2 & 0 & 0 & 0 \\ 1/3 & 2/3 & 0 & 0 & 0 & 0 \\ 1/2 & 0 & 1/2 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 1/3 & 0 & 1/3 & 1/3 \\ 0 & 0 & 0 & 1/2 & 0 & 1/2 \end{pmatrix}$$

(a) Draw a transition diagram for the Markov chain.

(b) If $X_0 = 5$, what is the distribution of X_2 (write it as a vector)?

(c) What are the communicating classes? For each communicating class, determine if it is transient or recurrent and whether it is periodic or aperiodic.

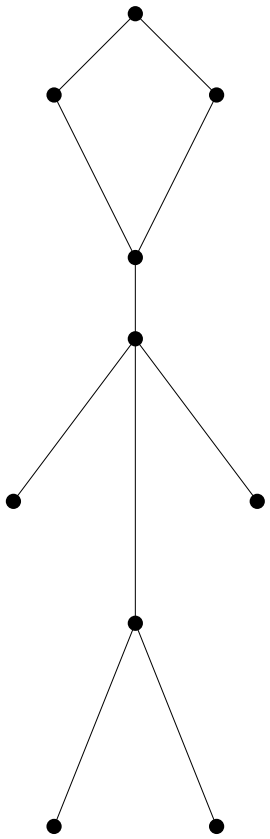
(d) Find a stationary distribution with $\pi_6 = 0$ (write it as a vector).

(e) If $X_0 = 1$, what is the expected time before the chain returns to 1?

9. [8 marks] A random walk on a graph has state space the nodes of the graph. From each node, it moves to a connected node, with equal probability for each connected node. (If x has degree d_x, and x, y are connected, then $P_{x,y} = 1/d_x$.)

   (a) Define: a Markov chain is reversible with respect to a measure π.

   (b) Show that the random walk on the graph below is reversible with respect to some distribution π, and find that π.



   (c) Find the asymptotic fraction of time the random walk spends at the topmost vertex in this graph.