Math 318 – homework 1 – due 2023-01-20

Problem 1. A coin is tossed until either two Tails appear successively, or until the fifth toss, whichever comes first. The outcome is the resulting sequence of coins.

(a) Write down the sample space, and determine the probability of each outcome in the sample space.
(b) For each $i \in \{2, 3, 4, 5\}$, let E_i be the event that the coin is tossed exactly i times. Determine $P(E_i)$ for each i.

Problem 2. An herpetology graduate student is sent to estimate the number of frogs in a pond. She captures 40 frogs, marks each with a dot of paint, and then releases them. The next day, she goes back and captures another sample of 50. She finds that 14 of the frogs were previously marked, and 36 unmarked.

Assuming that the frog population has size n, and that every frog is equally likely to be captured, determine the probability $L(n)$ that a sample of 50 frogs will contain exactly 14 marked ones.Show that the function $L(n)$ is increasing up to some n_*, and decreasing afterwards. Hint: when does the inequality $L(n)/L(n-1) \leq 1$ hold? Find the maximum likelihood estimate for n; that is the value n_* which maximizes $L(n)$.

Problem 3. (a) Compute the probability that a poker hand contains:
  (i) one pair ($aabcd$ with a, b, c, d) distinct face values; answer: 0.42)
  (ii) two pairs ($aabbc$ with a, b, c distinct face values; answer: 0.047)

(b) Poker dice is played by simultaneously rolling 5 dice. Compute the probabilities of the following outcomes:
  (i) one pair ($aabcd$ with a, b, c, d) distinct face values; answer: 0.46)
  (ii) two pairs ($aabbc$ with a, b, c distinct face values; answer: 0.23)

Problem 4. A coin is tossed $2n$ times. Let p_n be the probability that exactly half the outcomes are heads.

(a) Find a formula for p_n.
(b) Calculate p_{n+1}/p_n, and show that p_n is decreasing in n (i.e., $p_{n+1} < p_n$).
(c) We are interested in the asymptotics of p_n. We use the notation $a_n \sim b_n$ if $\lim a_n/b_n = 1$. Using Stirling's formula

$$n! \sim \sqrt{2\pi n}(n/e)^n,$$

prove that there is some α so that $p_n \sim \alpha/\sqrt{n}$, and find the value of α.

Problem 5. The number of ways to place n *distinguishable* balls in m urns is m^n, since each ball can be placed in any one of the m urns. The multinomial coefficient $\binom{n}{n_1,\ldots,n_m} = \frac{n!}{n_1!\ldots n_m!}$ counts the number of ways that n_i balls are in urn i for each $i = 1, 2, \ldots, m$, so when each ball is randomly assigned to an urn, the probability that n_i balls are in urn i, for each i, is equal to $\binom{n}{n_1,\ldots,n_m}m^{-n}$. Systems described by these probabilities are said to obey Maxwell–Boltzmann statistics.

(a) Suppose instead that the balls are *indistinguishable*; now we speak of Bose–Einstein statistics. When there are $m = 2$ urns, the number of ways of putting the n balls in the 2 urns is $n + 1$, because an outcome is specified by saying how many balls are in urn 1 and the possibilities are $\{0, 1, 2, \ldots, n\}$. For the case of general $m \geq 1$, how many ways are there to place n indistinguishable balls in m urns?
Hint: This is the number of ways to arrange $m-1$ barriers among a row of n balls, e.g., for $n = 7$ and $m = 3$ the configuration with $n_1 = 0, n_2 = 2, n_3 = 5$ is graphically described by $| \bullet\bullet | \bullet\bullet\bullet\bullet\bullet$.
(b) Indistinguishable particles are said to obey Fermi–Dirac statistics if all arrangements that have at most one ball per urn have the same probability, and those are the only arrangements. How many ways can n of these particles be put into m urns (assuming $m \geq n$

Problem 6. write a program in python (A Jupyter notebook may be convenient) that will do the following.

(a) Write a function `birthday(n)` that:
  (i) Generates a list containing n numbers uniformly distributed on $\{1, 2, \ldots, 365\}$ (think of this as the list of birthdays of n people, excluding leap year).

(ii) Returns 1 (or `True`) if there is at least one pair of people with coinciding birthdays (a "match") and 0 (or `False`) otherwise.

(b) For each n, from 2 to 60, run the function `birthday(n)` 1000 times, and compute the proportion $X(n)$ of the 1000 times in which there was a match.

(c) Let $Y(n)$ be the actual probability of a match:

$$Y(n) = 1 - \frac{365 \cdot 364 \cdots (365 - n + 1)}{365^n}.$$

In a single graph, plot of $X(n)$ and $Y(n)$ for $n \in [2, 60]$.

(d) Repeat the steps above for Martians. (Hint: The Martian year has 669 Martian days.)