Math 318 – homework 10 – due 2023-04-07

Problem 1. Find the stationary distribution (if it exists) of the following birth and death chains. In all cases the states are $\{0, 1, \dots\}$, and $P_{0,-1} = 0$ and $P_{0,1} = 1$.

(a) $P_{i,i+1} = \frac{1}{1+i}$ and $P_{i,i-1} = \frac{i}{i+1}$.
(b) $P_{i,i+1} = \frac{i}{1+i}$ and $P_{i,i-1} = \frac{1}{i+1}$.
(c) $P_{i,i+1} = P_{i,i-1} = \frac{1}{2}$ for $i < N$, and $p_{N,N} = 0$ and $P_{N,N-1} = 1$.

Problem 2. Consider the random walk on $\mathbb{N}$ with jump probabilities $p_{n,n+1} = \frac{n}{n+1}$ and $p_{n,n-1} = \frac{1}{n+1}$ for $n > 0$. We start the chain at $X_0 = 1$, and wish to find the probability that it never reaches 0. Let q_i be the probability of reaching 0 if we start at i.

(a) Write equations relating q_i to q_{i-1} and q_{i+1}.
(b) Let $a_n = \sum_{i<n} \frac{1}{i!}$. Show that a_n satisfy the equations above.
(c) If we know $\lim q_n = 1$, what is the value of q_n?
(d) Explain why $\lim q_n = 1$.

Problem 3. A reversible Markov chain on $\{0, 1, 2\}$ has transition matrix below, with some missing elements. Find the missing terms.

$$P = \begin{pmatrix} 1/2 & 1/3 & 1/6 \\ 1/3 & 0 & 2/3 \\ ? & ? & 0 \end{pmatrix}.$$

Problem 4. Alice, Bob and Carol play a game. At each round two of them participate (chosen randomly). They toss a coin, and the winner gets \$1 from the loser. If any of them runs out of money, they leave the game and do not participate any more. Initially Alice and Bob have \$2 and Carol has \$1. At the end, one winner has all the money.

(a) Find the probability that Carol ends up winning. (Hint: Once a player is eliminated, this is Gambler's ruin.)
(b) Find the probability that Carol is the first to be eliminated.

Problem 5. Simulate the following model for opinions. There are $N = 1000$ voters. Initially each has a different opinion. (The initial opinion of voter i is the number i.) At each step, pick randomly two voters x, y, and x copies the opinion of y (and forgets their previous opinion).

Run the process until there is only one opinion left. How many steps did that take? Repeat this several times to estimate the expected time.

(Note: `set(A)` for an array A may be useful for finding the number of distinct elements.)

Extra practice problems

Simulate a Branching process with Poisson offspring distribution for λ equal, slightly below and above 1. Investigate the survival probability to some large generation n, and the average size fot eh generations, bot overall and conditioned on survival.