Math 318 – homework 10 – due 2023-04-07

Problem 1. Find the stationary distribution (if it exists) of the following birth and death chains. In all cases the states are $\{0, 1, \dots\}$, and $P_{0,-1} = 0$ and $P_{0,1} = 1$.

(a) $P_{i,i+1} = \frac{1}{1+i}$ and $P_{i,i-1} = \frac{i}{i+1}$.

(b) $P_{i,i+1} = \frac{i}{1+i}$ and $P_{i,i-1} = \frac{1}{i+1}$.

(c) $P_{i,i+1} = P_{i,i-1} = \frac{1}{2}$ for $i < N$, and $p_{N,N} = 0$ and $P_{N,N-1} = 1$.

Solution.

(a) Detailed balance gives $\pi_i = \frac{i+1}{i^2}\pi_{i-1}$, from which we get $\pi_n = \frac{n+1}{n!}\pi_0$. Since

$$1 = \sum \pi_n = \sum \frac{n+1}{n!}\pi_0 = 2e\pi_0,$$

we have $\pi_0 = \frac{1}{2e}$ and $\pi_n = \frac{n+1}{2en!}$.

(b) Here $\pi_i = \frac{(i-1)(i+1)}{i}\pi_i$, so π_i is increasing in i and cannot add up to 1, so there is no stationary distribution.

Problem 2. Consider the random walk on $\mathbb{N}$ with jump probabilities $p_{n,n+1} = \frac{n}{n+1}$ and $p_{n,n-1} = \frac{1}{n+1}$ for $n > 0$. We start the chain at $X_0 = 1$, and wish to find the probability that it never reaches 0. Let q_i be the probability of reaching 0 if we start at i.

(a) Write equations relating q_i to q_{i-1} and q_{i+1}.

(b) Let $a_n = \sum_{i<n} \frac{1}{i!}$. Show that a_n satisfy the equations above.

(c) If we know $\lim q_n = 1$, what is the value of q_n?

(d) Explain why $\lim q_n = 1$.

Solution.

(a) Using first step analysis, we have

$$q_i = \frac{i}{i+1}q_{i+1} + \frac{1}{i+1}q_{i-1}.$$

We also have $q_0 = 0$.

(b) We have $a_{n+1} = a_{n-1} + \frac{1}{(n-1)!} + \frac{1}{n!}$. Therefore

$$\begin{aligned}\frac{n}{n+1}a_{n+1} + \frac{1}{n+1}q_{n-1} &= \frac{n}{n+1}a_{n-1} + \frac{n}{n+1}(1/(n-1)!) + 1/n!) + \frac{1}{n+1}q_{n-1} \\ &= a_{n-1} + \frac{n}{n+1}(n/n! + 1/n!) \\ &= a_{n-1} + \frac{1}{(n-1)!} = a_n.\end{aligned}$$

(c) The general solution to the recursion for q is $q_n = \alpha a_n + \beta$ for some α, β. In order to have $q_0 = 0$ we have $\beta = 0$. In order to have $q_n \to 1$ we need $\alpha = 1/e$, so the solution is $q_n = \frac{a_n}{e}$.

(d) The chain is a random walk on the integers with a bias to the right, that only gets stronger as it moves away from 0. Since the random walk with fixed bias tends to infinity, the walk with stronger bias should also tend to infinity with positive probability. If we start at n, the probability of returning to 0 must be very small.

Problem 3. A reversible Markov chain on $\{0, 1, 2\}$ has transition matrix below, with some missing elements. Find the missing terms.

$$P = \begin{pmatrix} 1/2 & 1/3 & 1/6 \\ 1/3 & 0 & 2/3 \\ ? & ? & 0 \end{pmatrix}.$$

Solution. If the chain is reversible then $p_{01}p_{12}p_{20} = p_{02}p_{21}p_{10}$. (this is the cycle condition.) This gives $\frac{1}{3}\frac{2}{3}p_{20} = \frac{1}{6}p_{21}\frac{2}{3}$. Therefore $2p_{20} = p_{21}$ and since P is stochastic we must have $p_{20} = 1/3$ and $p_{21} = 2/3$.

Problem 4. Alice, Bob and Carol play a game. At each round two of them participate (chosen randomly). They toss a coin, and the winner gets $1 from the loser. If any of them runs out of money, they leave the game and do not participate any more. Initially Alice and Bob have $2 and Carol has $1. At the end, one winner has all the money.

(a) Find the probability that Carol ends up winning. (Hint: Once a player is eliminated, this is Gambler's ruin.)

(b) Find the probability that Carol is the first to be eliminated.

Solution.

(a) This is 1/5. More generally, if Carol has $k, and the total amount of money in the game is $n, then the probability that Carol is the last winner is k/n. This is just the gambler's ruin problem for Carol. Some rounds she sit out and these have no effect. When she plays, she is equally likely to win or lose a dollar. So the probability she reaches n before 0 is k/n.

The technical term for this is a lazy random walk: a walk that sometimes just stays in place.

(b) This has no simple solution in general. For a small setting such as the above, we can use first step analysis. Let q_{abc} be the probability that Carol is first to be eliminated if the players have a, b, c respectively. There are 6 states we care about. In each one there are 6 possible steps, since one player sits out, one wins and one loses. The equations are

$$\begin{aligned}
q_{221} &= \frac{1}{6}(q_{311} + q_{131} + q_{122} + q_{212} + 1 + 1) \\
q_{212} &= \frac{1}{6}(q_{311} + q_{122} + q_{221} + q_{113} + 0 + 0) \\
q_{122} &= \frac{1}{6}(q_{221} + q_{212} + q_{113} + q_{131} + 0 + 0) \\
q_{311} &= \frac{1}{6}(q_{221} + q_{212} + 0 + 1 + 0 + 1) \\
q_{131} &= \frac{1}{6}(q_{221} + q_{122} + 0 + 1 + 0 + 1) \\
q_{113} &= \frac{1}{6}(q_{122} + q_{212} + 0 + 0 + 0 + 0)
\end{aligned}$$

The 1 or 0 terms come from events where either Carol is eliminated, or someone else is. We can use symmetry to see that $q_{212} = q_{122}$ and $q_{131} = q_{311}$. This gives 4 equations in 4 variables:

$$\begin{aligned}
q_{221} &= \frac{1}{6}(2q_{311} + 2q_{212} + 2) \\
q_{212} &= \frac{1}{6}(q_{311} + q_{212} + q_{221} + q_{113}) \\
q_{311} &= \frac{1}{6}(q_{221} + q_{212} + 2) \\
q_{113} &= \frac{1}{6}(2q_{212})
\end{aligned}$$

The solution of these gives $q_{221} = 23/41$.

Problem 5. Simulate the following model for opinions. There are $N = 1000$ voters. Initially each has a different opinion. (The initial opinion of voter i is the number i.) At each step, pick randomly two voters x, y, and x copies the opinion of y (and forgets their previous opinion).

Run the process until there is only one opinion left. How many steps did that take? Repeat this several times to estimate the expected time.

(Note: `set(A)` for an array A may be useful for finding the number of distinct elements.)