Math 318 – homework 11 – undue

Problem 1. A bus is getting slowly filled with groups of 1,2,3 people, until it is full. To model this, let X_n be independent random variables, uniform on $\{1, 2, 3\}$, and let $S_n = \sum_{i \le n X_i}$ be the total number of people in the first n groups. The bus has capacity N. Once a group shows up for which there is no space, the bus leaves. Therefore, let k be the minimal such that $S_k > N$. We are interested in the size of the lat group: X_k.

(a) Let a_N be the probability that $S_k = N$ (the bus is exactly full). Write a recursion giving a_N in terms of other a_m. (hint: consider X_1).
(b) Find expressions for $P(S_k = N + i)$ in terms of a_m for $i = 1, 2, 3$.
(c) Explain why $a_N \to 1/2$, and conclude that $P(X_k = i) \to i/6$ for $i = 1, 2, 3$.

Problem 2. Consider the graph where every vertex on level k has two children at level $k+1$, with no other edges. At level 0 there is one vertex, so at level k there are 2^k. Prove that the random walk on this graph is transient, and find the probability that the walk returns to level 0 if it start there. (Hint: conside what level the walk is on.)

Problem 3. Consider the following Markov chain with state space $\{0, 1, \dots, 2^n - 1\}$. From state k it jumps to either $2k \bmod 2^n$ or $2k + 1 \bmod 2^n$ with probability 1/2 each. Find the stationary distribution for the chain, and show that the n state transition probabilities are $P^n_{ij} = \pi_j$ for every i, j.

Problem 4. Let P be a doubly stochastic matrix: row and column sums are all 1. Show that the Markov chain with transition matrix P has stationary distribution uniform over the states.

Problem 5. Consider a version of the bus problem, with groups that have distribution $Geom(.1)$. Run a simulation of this with N=1000, and estimate the distribution of X_k based on the simulation. Compare the resulting distribution to the $Geom(.1)$, and to a distribution with $p(k) = c0.9^k$ for some c.

Problem 6. This problem involves the Bienayme-Galton-Watson branching process. Let each individual have a random number of children which is Poi(λ) for some λ.

(a) For each $\lambda \in \{0.9, 1, 1.1, 2\}$, simulate the process starting with one individual in generation 0, for 100 generations. Repeat this 1000 times for each λ. How many of the processes died out for ech λ?
(b) For each λ, plot the average over the experiments of the size of each generation for each λ.
(c) For each λ, plot the average over the experiments of the size of each generation for each λ, when taking into the average for generation k only those that survived to generation k at least.