Math 318 – homework 3 – due 2023-02-03

Problem 1. Two hockey teams, A and B play a series of games, until one of the teams wins 4 games. Suppose team A has probability p of winning each game, and games are independent. Let X be the total number of games that are played.

(a) Find the probability mass function of X.
(b) What is the probability that team A wins the series conditioned on $X = 4$?
(c) What is the probability that team A wins the series conditioned on $X = 7$? (Simplify your expressions as much as possible.)

Problem 2. A fair (6-sided) die is rolled four times.

(a) Let Y denote the number of distinct results. Find the probability mass function and expectation of Y.
(b) Let Z denote the minimal result out of the 4 throws. Find the probability mass function and expectation of Z.

Problem 3. This problem investigates the similarity between the geometric and exponential random variables observed last week. Let Y be a geometric random variable with parameter p, so that Y represents the trial number of the first success in a sequence of independent Bernoulli trials. Suppose the trials occur at times $\delta, 2\delta, \dots$, and that δ and p are both very small. Let $\lambda = p/\delta$. At time t, about t/δ trials have taken place.

(a) Compute $P(Y > m)$, which represents the probability that no success has been observed by time $t = m\delta$.
(b) Show that the probability that no success has been observed by time t converges to $e^{-\lambda t}$ as $p, \delta \to 0$ with $\lambda = p/\delta$ fixed.
(c) Conclude that the time of the first success is approximately an exponential random variable with parameter λ.

Problem 4. A binary message either 0 or 1 is transmitted by wire. However, data sent over the wire is subject to channel noise disturbance. If x is the value sent (either 0 or 1), then the value received at the other end is $R = x + N$, where N represents the noise. Assume that N is a normal random variable with mean $\mu = 0$ and variance $\sigma^2 = 0.04$. Assume that a message sent is equally likely to be 0 or 1. When the message is received the receiver decodes it according to the following rule: If $R \leq \frac{1}{2}$ she concludes the message is 0, and otherwise concludes it is 1. What is the probability that the message is received correctly?

Problem 5. The number of murders in Gotham on any given week is assumed to be Poisson with unknown mean λ, with different weeks independent.

(a) We observe there were 2 murders one week. What is the value of λ for which this is most likely?
(b) We observe for a second week, and there is 1 murder. What is the value of λ for which this pair of observations is most likely?
(c) Generalize the above to observations $a_1, \dots, a_k$ over k weeks.

Problem 6. An airline books passengers for a flight on an airplane with 420 seats. From experience, the airline knows that each passenger has probability $p = \frac{1}{50}$ of missing the flight. Assume these events are independent. As such, the airline takes a risk and sells 430 tickets for the flight.

(a) Using python, compute the probability that more than 420 passenger show up.
(b) Use the Poisson approximation to compute an approximation to this probability.
(c) Simulate the number of no-shows for an overbooked flight 50000 times. (You can use a function that returns a binomial random variable.) Plot a histogram of the fraction of times there were k no-shows, and the Poisson p.m.f. on the same graph.
(d) Simulated the number of no-shows 50000 times, and define X_n = number of overfull flights in the first n simulated bookings. Then X_n/n is the running proportion of overbooked flights. Plot X_n/n. What happens to it as n gets large?
Note: When calculating X_n, do not simulate n new bookings for every n. Simulate 50000 flights, and then for every n, calculate X_n based on the first n of these.

II. Recommended problems:

(a) Chapter 2: 16, 23, 27, 31, 38, 39, 40, 49.

(b) Three sections of a class contain 70, 80 and 150 students respectively.

  (i) If a section is chosen at random, what is its expected size?

  (ii) If a student is chosen at random, what is the expected size of their section? Why are these different?