Math 318 – homework 4 solutions

Problem 1. A particle of mass 1g has a random velocity X that is uniformly distributed between 3cm/s and 8cm/s.

(a) Find the cumulative distribution function of the particle's kinetic energy $T = \frac{1}{2}X^2$.
(b) Find the probability density function of T.
(c) Find the mean of T.

Solution.

(a) Let F be the CDF of the kinetic energy. Then

$$F(t) = P(T \leq t) = P(X \leq \sqrt{2t}).$$

This is $\frac{\sqrt{2t}-3}{5}$ when $\sqrt{2t} \in [3,8]$ and is 0 or 1 for smaller or larger t. In summary,

$$F(t) = \begin{cases} \frac{\sqrt{2t}-3}{5} & t \in [9/2, 32] \\ 0 & t < 9/2 \\ 1 & t > 32. \end{cases}$$

(b) The PDF is $F'(t) = \frac{1}{5\sqrt{2t}}$ on $[9/2, 32]$, and 0 outside.

(c) The mean is

$$E[T] = \int_{9/2}^{32} \frac{t}{5\sqrt{2t}} dt = \frac{97}{6}.$$

Using the law of the unconcious statistician, this can also be found directly:

$$E[T] = E[X^2/2] = \int_3^8 \frac{x^2}{2}\frac{dx}{5}.$$

Problem 2. Stanislaw is collecting coupons. Each day he receives randomly one of n distinct coupons with equal probabilities (independently of other days).

(a) Let T be the number of days it takes Stanislaw to obtain a complete set. Explain why T can be written as a sum of n independent Geometric random variables (and say what their parameters are).
(b) Compute the expected value of T. (Use the fact that the expectation of a sum of random variables is the sum of the expectations.)

Solution.

(a) Let T_i be the number of days until Stanislaw has i different coupons. We let $T_0 = 0$, and clearly $T_1 = 1$. We are intersted in $T = T_n$. Let $X_i = T_i - T_{i-1}$, so $T = X_1 + X_2 + \cdots + X_n$. Once Stanislaw has $i-1$ coupons, each day there is probability $\frac{n-i+1}{n}$ of getting a new coupon. These attempts to get a new coupon are independent until the next succecss, so X_i is Exp($\frac{n-i+1}{n}$) with expectation $\frac{n}{n-i+1}$.

(b) Using additivity of expectations,

$$E[T] = \sum_{i=1}^n E[X_i] = \sum_{i=1}^n \frac{n}{n-i+1}.$$

Therefore $E[T] = \frac{n}{n} + \frac{n}{n-1} + \cdots + \frac{n}{2} + \frac{n}{1} \approx n \log n$.

Problem 3. The time T (in hours past noon) until the arrival of the first taxi has Exp(6) distribution, and the time B until first bus is independent with Exp(4) distribution.

(a) Write down the joint probability density function of T and B. (Pay attention to when it is 0.)
(b) Find the probability that the first taxi arrives before the first bus.
(c) If you arrive at noon and take the first bus or taxi (whichever arrives first), what is the distribution of your waiting time? (Give the PDF, and/or the CDF.) Does it have a name? (Hint: denote $X = \min(T, B)$, and find $P(X > y)$.)

Solution.
(a) From independence, this is $f(t,b) = 6e^{-6t} \cdot 4e^{-4b} = 24e^{-6t-4b}$ for $b, t \geq 0$.
(b) This probability is

$$\iint_{0<t<b} f(t,b)dtdb = \int_0^\infty \left(\int_t^\infty f(t,b)db\right) dt = \int_0^\infty 6e^{-10t}dt = \frac{6}{10}.$$

(c) Let X be the waiting time. Then $P(X > x) = P(T > x, B > x) = P(T > x)P(B > x) = e^{-6x}e^{-4x}$. Therefore $P(X > x) = e^{-10x}$ and X is Exp(10).

Note. The same calculation shows that minimum of independent Exp(λ_1) and Exp(λ_2) is Exp($\lambda_1 + \lambda_2$), and a similar result holds for the minimum of several independent exponentials.

Problem 4. If X, Y are independent $N(0,1)$ random variables, what is the distribution of $R = \sqrt{X^2 + Y^2}$? Does it have a name?

Solution. The joint density of X, Y is $\frac{1}{2\pi}e^{-(x^2+y^2)/2}$. Let $R = \sqrt{X^2 + Y^2}$, so the CDF of R is

$$F_R(s) = \iint_{R\leq s} \frac{1}{2\pi}e^{-(x^2+y^2)/2}dxdy.$$

We can change to polar coordinates to get

$$F_R(s) = \int_0^{2\pi} d\theta \int_0^s \frac{1}{2\pi}e^{-r^2} r\, dr,$$

where the r term is the Jacobian in the change of variable. By a change of variable $t = r^2$, and integrating over θ, this becomes

$$F_R(s) = \int_0^{s^2} e^{-t}\frac{dt}{2} = \frac{1 - e^{-s^2}}{2}.$$

The pdf is therefore $F_R'(s) = se^{-s^2}$.

Problem 5. This question considers uniform random points on the unit disc $x^2 + y^2 \leq 1$.
(a) A point (X, Y) is uniformly chosen in the unit disc. Find the CDF and PDF of its distance from the origin $R = \sqrt{X^2 + Y^2}$.
(b) Compute the expected distance from the origin.
(c) Determine the marginal PDF of X and Y.
(d) Are X and Y independent? (Justify your claims).
(e) One way to generate uniform random points on this disc is to first generate uniform random points on the square $[-1, 1] \times [-1, 1]$ by selecting their coordinates independently, and ignoring points that lie outside the unit disc. To visualize this, generate 10000 uniform random points on the square and and create a scatter-plot of the points inside the disc, discarding the points outside the disc.
(f) Another way to represent points in the plane is via polar coordinates $(R\cos\Theta, R\sin\Theta)$, with $R \in [0, 1]$ and $\Theta \in [0, 2\pi]$. We might try naively to generate uniform random points in the circle by first generating a random radius R uniformly in $[0, 1]$, and then by generating a random angle Θ uniformly in $[0, 2\pi]$. Generate 10000 such random pairs (R, Θ) and create a scatter-plot of the resulting points in the plane. Does this appear uniformly random? Compare the two plots.
(g) Let the density of uniformly random points in the circle with respect to polar coordinates is the function $f(r, \theta)$, so that if A is a subset of the disc then

$$\frac{\text{area}(A)}{\pi} = \iint_A f(r,\theta)drd\theta.$$

Using your knowledge of multivariate calculus, what must f be?

Solution.

(a) The CDF: $F(r) = P(R \le r) = r^2$ for $r \in [0,1]$ (and 0 or 1 for smaller or larger r). The PDF is $2r$ for $r \in [0,1]$.

(b) $E[R] = \int r f_R(r) dr = \int_0^1 r \cdot 2r dt = \frac{2}{3}$.

(c) Marginal distribution of X: Since $f(x,y) = \frac{1}{\pi}$ when $y \in [-\sqrt{1-x^2}, \sqrt{1-x^2}]$, we get

$$f_X(x) = \frac{2}{\pi}\sqrt{1-x^2} \qquad \text{for } x \in [-1,1].$$

The marginal of Y is the same.

(d) X and Y are not independent. For example, $P(X > 0.9) > 0$, and the same for $Y > 0.9$, but $P(X > 0.9 \text{ and } Y > 0.9) = 0$.

(e,f) See Jupyter notebook. The plots are clearly different.

(g) Since the density is $\frac{1}{\pi}$ with respect to x, y, the change of variable to polar coordinates gives $f(r,\theta) = \frac{2r}{2\pi}$ for $r \le 1$ and $\theta \in [0, 2, \pi]$. This is since they are independent, with density $2r$ for R and $1/2\pi$ for Θ.

Problem 6. (a) Use Python to sample a standard normal random variable 10,000 times independently and plot the running average: If the variables are $X_1, X_2, \dots$ make a plot of $\frac{X_1+\dots+X_n}{n}$ for $n \le 10000$.

(b) Repeat the same exercise for Exponential variables with parameter $\lambda = 2$.

(c) Repeat the same for Cauchy random variables. (Recall: a Cauchy variable is the x-axis intersection of a line through $(0,1)$ with a random direction θ uniform in $[0,\pi]$, so that $X = \tan\theta$.)

(d) Does each of the plots seem to converge? what are the limits if so?

Solution.

(a,b,c) see notebook.

(d) The first two converge to $0 = E[N(0,1)]$ and $1/2 = E[\text{Exp}(2)]$. This demonstrates the law of large numbers. The third does not converge, since the Cauchy variable has infinite expectation.

Extra practice problems

A. Chapter 2: 37, 41, 43, 50,51,57.

B. Chapter 5: 2,18.

C. If you are familiar with quantum mechanics:
Consider a quantum mechanical system in state $\psi \in H$, where H is the hilbert space of complex functions with inner product $\langle \phi, \psi \rangle = \int \phi(x)\overline{\psi(x)}dx$. We may assume ψ is normalized, so that $\langle psi, \psi = \int |\psi(x)|^2 dx = 1$. Observables of the system (such as position $\mathcal{X}$ or momentum $\mathcal{P}$ of a particle) are represented by self-adjoint linear operators on H. The expected value of an observable A is given by $\langle \psi, A\psi \rangle$. The standard deviation $\sigma(A)$ of a measurement of the observable A is given by $\sigma(A)^2 = \langle \psi, A^2\psi \rangle - \langle \psi, A\psi \rangle^2$. It is a general mathematical theorem (see Lemma 6.1 of E. Prugovecki, Quantum Mechanics in Hilbert Space, 1971) that for any self-adjoint linear operators A, B, with commutator $[A,B] = AB - BA$ we have

$$\langle \psi, A^2\psi \rangle \langle \psi, B^2\psi \rangle \ge \frac{|\langle \psi, [A,B]\psi \rangle|^2}{4}.$$

The commutator of the position and momentum operators is $[\mathcal{X}, \mathcal{P}] = \hbar i$.
Use the above to prove the **uncertainty principle**: $\sigma(\mathcal{X})\sigma(\mathcal{P}) \ge \hbar/2$.